MATH 614

Dynamical Systems and Chaos

# Lecture 30: Neutral fixed points (continued).

# Classification of periodic points

Let $U \subset \mathbb{C}$ be a domain and $F : U \to \mathbb{C}$ be a holomorphic function. Suppose that $F(z_0) = z_0$ for some $z_0 \in U$. The fixed point $z_0$ is called

- attracting if $|F'(z_0)| < 1$;
- repelling if $|F'(z_0)| > 1$;
- neutral if $|F'(z_0)| = 1$.

Now suppose that $F^n(z_1) = z_1$ for some $z_1 \in U$ and an integer $n \geq 1$. The periodic point $z_1$ is called

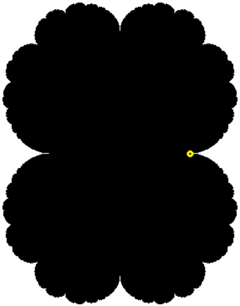
- attracting if $|(F^n)'(z_1)| < 1$;
- repelling if $|(F^n)'(z_1)| > 1$;
- neutral if $|(F^n)'(z_1)| = 1$.

The multiplier $(F^n)'(z_1)$ is the same for all points in the orbit of $z_1$ (in particular, all these points are of the same type as $z_1$). Moreover, the multiplier is preserved under any holomorphic change of coordinates.

# Neutral fixed points

*Example.* • $F(z) = z + z^2$.

The map has a fixed point at 0, which is neutral: $F'(0) = 1$. The set $D_0$ of all points $z$ satisfying $F^n(z) \to 0$ as $n \to \infty$ is open and connected.

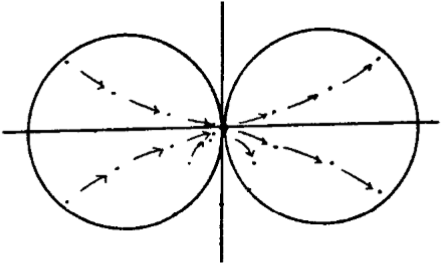


The fixed point 0 is one of the cusp points at the boundary of $D_0$. The others correspond to eventually fixed points.

# Neutral fixed points

**Proposition** Suppose a function $F$ is holomorphic at 0 and satisfies $F(0) = 0$, $F'(0) = 1$, $F''(0) = 2$ so that $F(z) = z + z^2 + O(|z|^3)$ as $z \to 0$.

Then there exists $\mu > 0$ such that **(i)** all points in the disc $D_- = \{z \in \mathbb{C} : |z + \mu| < \mu\}$ are attracted to 0; and **(ii)** all points in the disc $D_+ = \{z \in \mathbb{C} : |z - \mu| < \mu\}$ are repelled from 0.



*Proof:* We change coordinates using the function $H(z) = 1/z$, which maps the discs $D_-$ and $D_+$ onto halfplanes $\operatorname{Re} z < -1/(2\mu)$ and $\operatorname{Re} z > 1/(2\mu)$.

The function $F$ is changed to $G(z) = 1/F(1/z)$. Since $F(z) = z + z^2 + O(|z|^3)$ as $z \to 0$, it follows that

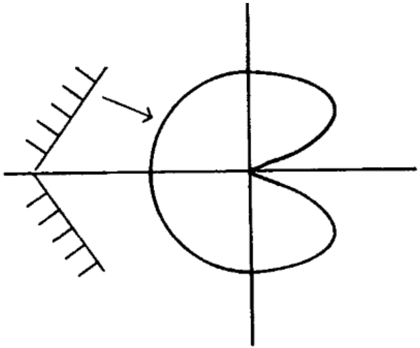
$$\begin{aligned} F(1/z) &= z^{-1} + z^{-2} + O(|z|^{-3}) \\ &= z^{-1}\big(1 + z^{-1} + O(|z|^{-2})\big) \quad \text{as} \quad z \to \infty. \end{aligned}$$

Then

$$\begin{aligned} G(z) &= z\big(1 + z^{-1} + O(|z|^{-2})\big)^{-1} \\ &= z\big(1 - z^{-1} + O(|z|^{-2})\big) = z - 1 + O(|z|^{-1}). \end{aligned}$$

If $\mu$ is small enough, then the halfplane $\operatorname{Re} z < -1/(2\mu)$ is invariant under the map $G$ while the halfplane $\operatorname{Re} z > 1/(2\mu)$ is invariant under $G^{-1}$.

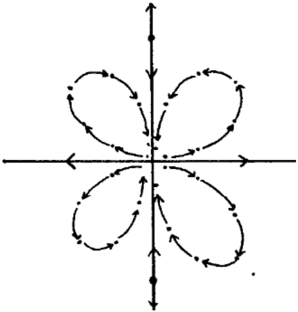
In the proof of the proposition, we could use wedge-shaped regions instead of halfplanes. This would allow to extend basins of attraction from discs to cardioid-shaped regions.

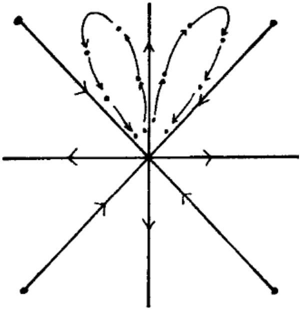


In the case not all points near 0 are attracted to 0, the set of points that are attracted is locally a simply connected domain with 0 on its boundary. This domain is called the **attracting petal** of the fixed point 0. Similarly, there is also the **repelling petal** of 0.

# More types of neutral fixed points

$F_1(z) = z + z^3$ $\qquad$ $F_2(z) = z + z^5$





In the first example, there are two attracting and two repelling petals of the fixed point 0. In the second example, there are 4 attracting and 4 repelling petals.

# Siegel discs

**Theorem (Siegel)** Let $F$ be a holomorphic function at $z_0$ such that $F(z_0) = z_0$ and $F'(z_0) = e^{2\pi i\alpha}$, where $\alpha$ is irrational. Suppose that $\alpha$ is not very well approximated by rational numbers, namely, $|\alpha - p/q| > aq^{-b}$ for some $a, b > 0$ and all $p, q \in \mathbb{Z}$. Then there is a neighborhood $U$ of $z_0$ on which the function $F$ is analytically conjugate to the irrational rotation $L(z) = e^{2\pi i\alpha}z$.

The domain $U$ is called a **Siegel disc**.

In the case $\alpha$ is well approximated by rational numbers, it can happen that the fixed point $z_0$ is a limit point of other periodic points of the map $F$.

*Idea of the proof:* We are looking for a map $h$ of the form $h(z) = z + \sum_{i=2}^{\infty} c_i z^i$, where $c_i$ are unknown coefficients. Let $F(z) = \lambda z + \sum_{i=2}^{\infty} a_i z^i$ be the Taylor expansion of $F$, $\lambda = e^{2\pi i\alpha}$.

The condition $F \circ h = h \circ L$ holds if

$$\lambda h(z) + \sum_{i=2}^{\infty} a_i (h(z))^i = \lambda z + \sum_{i=2}^{\infty} c_i \lambda^i z^i$$

or, equivalently,

$$\sum_{i=2}^{\infty} (\lambda^i - \lambda) c_i z^i = \sum_{i=2}^{\infty} a_i (h(z))^i.$$

From this equality of formal power series we can recursively determine all coefficients $c_i$ provided that $\lambda$ is not a root of unity. For example, $c_2 = a_2/(\lambda^2 - \lambda)$. In general, $c_i = K_i(\lambda, a_2, \ldots, a_i, c_2, \ldots, c_{i-1})/(\lambda^i - \lambda)$ for some polynomial $K_i$.

The Siegel disc exists if and only if the radius of convergence for the power series $h(z)$ is positive.