

MATH 614

Dynamical Systems and Chaos

Lecture 31:

The quadratic maps.

The Mandelbrot set.

The Julia set

Suppose $P : U \rightarrow U$ is a holomorphic map, where U is a domain in \mathbb{C} , the entire plane \mathbb{C} , or the Riemann sphere $\overline{\mathbb{C}}$.

Informally, the Julia set of P is the set of points where iterates of P exhibit sensitive dependence on initial conditions (chaotic behavior). The Fatou set of P is the set of points where iterates of P exhibit regular, stable behavior.

Definition. The **Julia set** $J(P)$ of P is the closure of the set of repelling periodic points of P .

Example. $Q_0 : \mathbb{C} \rightarrow \mathbb{C}$, $Q_0(z) = z^2$.

0 is an attracting fixed point. The other periodic points are located on the unit circle $|z| = 1$. All of them are repelling.

Any point of the form

$$\exp\left(2\pi i \frac{m}{n}\right),$$

where m, n are integers and n is odd, is periodic. Hence periodic points are dense in the unit circle.

Thus $J(Q_0) = \{z \in \mathbb{C} : |z| = 1\}$.

Quadratic family

The quadratic family $Q_c : \mathbb{C} \rightarrow \mathbb{C}$, $c \in \mathbb{C}$,
 $Q_c(z) = z^2 + c$.

Theorem $J(Q_{-2}) = [-2, 2]$.

Proof: The map $H(z) = z + z^{-1}$ is holomorphic on $R = \{z \in \mathbb{C} : |z| > 1\}$. It maps R onto $\mathbb{C} \setminus [-2, 2]$ in a one-to-one way (a conformal map). Also, H maps each of the semicircles

$$\{e^{i\phi} \mid 0 \leq \phi \leq \pi\} \quad \text{and} \quad \{e^{i\phi} \mid -\pi \leq \phi \leq 0\}$$

homeomorphically onto $[-2, 2]$.

Finally, $H(Q_0(z)) = Q_{-2}(H(z))$ for all $z \neq 0$.

Quadratic family

The quadratic family $Q_c : \mathbb{C} \rightarrow \mathbb{C}$, $c \in \mathbb{C}$,
 $Q_c(z) = z^2 + c$.

The dynamics of Q_c depends on properties of the post-critical orbit $0, Q_c(0), Q_c^2(0), \dots$

Theorem (Fundamental Dichotomy)

For any $c \in \mathbb{C}$,

either the post-critical orbit $0, Q_c(0), Q_c^2(0), \dots$ escapes to ∞ , in which case the Julia set $J(Q_c)$ is a Cantor set,

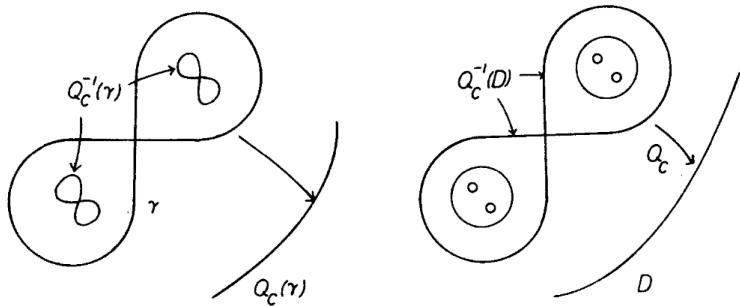
or the post-critical orbit is bounded, in which case the Julia set $J(Q_c)$ is connected.

Lemma If $|z| > \max(|c|, 2)$ then $|Q_c^n(z)| \rightarrow \infty$ as $n \rightarrow \infty$.

Suppose γ_0 is a simple closed curve in \mathbb{C} and D_0 is the domain bounded by γ_0 . Let $\gamma_1 = Q_c^{-1}(\gamma_0)$ and $D_1 = Q_c^{-1}(D_0)$.

Then γ_1 is the boundary of D_1 .

If $c \in D_0$ then γ_1 is a simple closed curve. If $c \in \gamma_0$ then γ_1 is a "figure eight". Otherwise γ_1 consists of two curves.



If D_0 is a large disk then $D_1 = Q_c^{-1}(D_0)$ is contained in D_0 . Hence $D_0 \supset D_1 \supset D_2 \supset \dots$, where $D_{n+1} = Q_c^{-1}(D_n)$, $n \geq 0$.

Note that the set $K(Q_c) = \bigcap_{n=0}^{\infty} D_n$ does not depend on the choice of D_0 .

Now either $c \in D_n$ for all n , in which case each D_n is a simply connected domain so that $K(Q_c)$ is also connected. In this case, $J(Q_c)$ is the boundary of $K(Q_c)$.

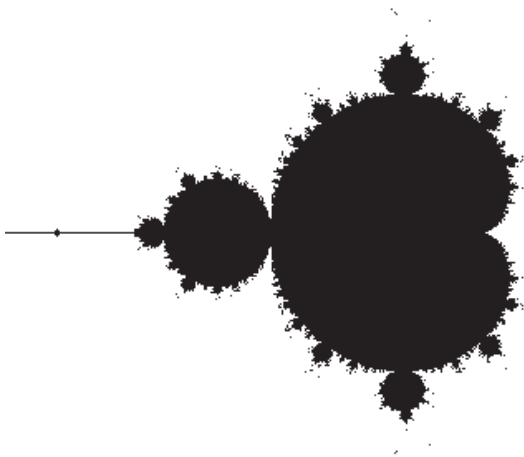
Otherwise $c \in D_n \setminus D_{n+1}$ for some n , in which case D_{n+k} has 2^{k-1} connected components, $k = 0, 1, 2, \dots$, so that $K(Q_c)$ is a Cantor set. In this case, $J(Q_c) = K(Q_c)$.

The Mandelbrot set

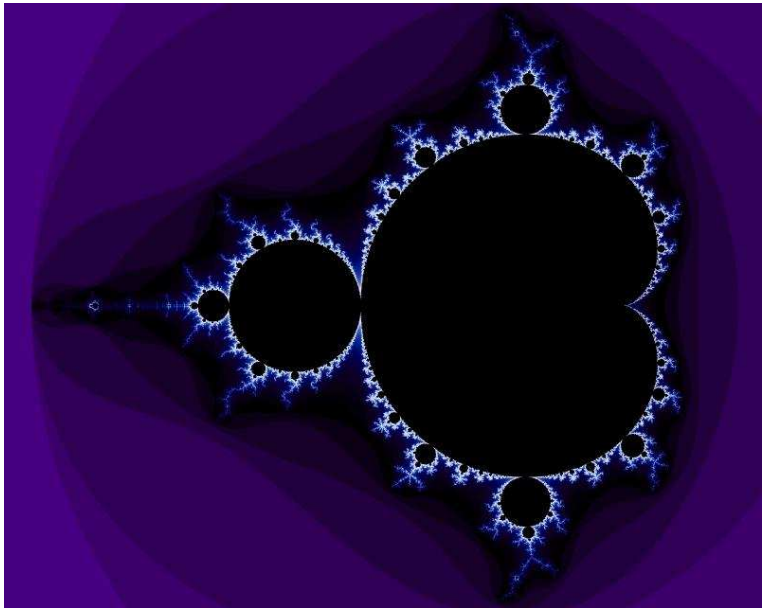
Definition. The **Mandelbrot set** \mathcal{M} is the set of all $c \in \mathbb{C}$ such that $|Q_c^n(0)| \not\rightarrow \infty$ as $n \rightarrow \infty$.

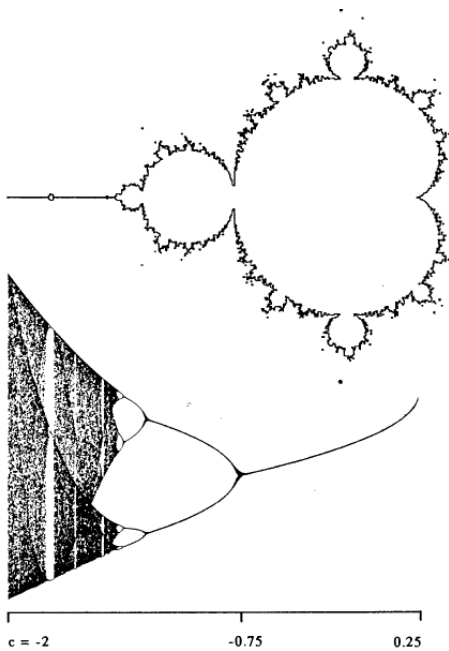
$c \in \mathcal{M}$ if and only if the Julia set $J(Q_c)$ is connected.

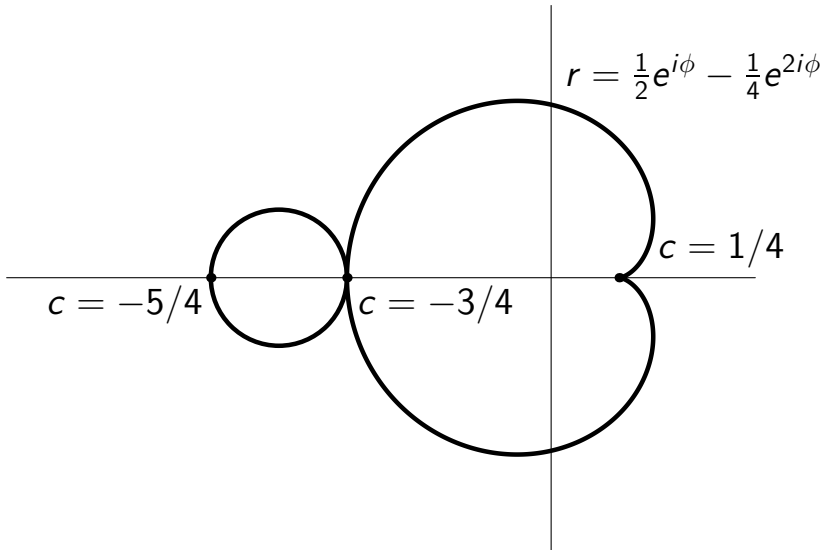
The Mandelbrot set \mathcal{M} is the bifurcation diagram for the quadratic family.



The Mandelbrot set





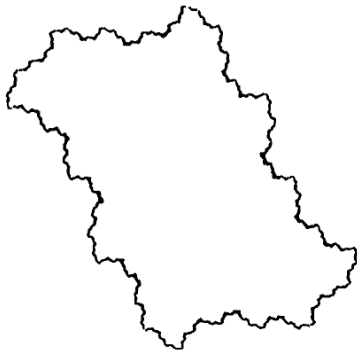


The main cardioid and the period 2 bulb

Theorem 1 For any c within the main cardioid, the Julia set $J(Q_c)$ is a simple closed curve.

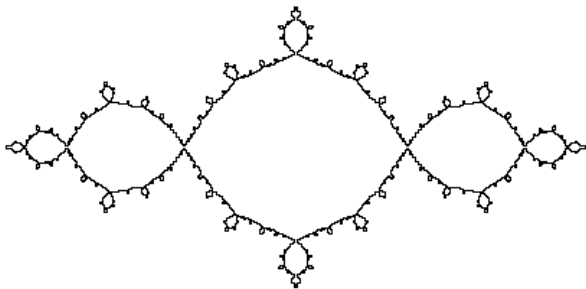
The region bounded by this curve is the basin of attraction of an attracting fixed point.

If $c \notin \mathbb{R}$ then $J(Q_c)$ contains no smooth arc.



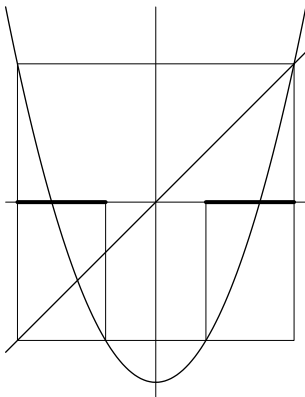
$$J(Q_c), \quad c = i/2$$

Theorem 2 For any c within the period 2 bulb, $J(Q_c)$ is the closure of countably many simple closed curves. The interior of $K(Q_c)$ has countably many connected components and coincides with the basin of attraction of an attracting periodic orbit of period 2.



$$J(Q_c), \quad c = -1$$

Theorem 3 For any $c \notin \mathcal{M}$, the Julia set $J(Q_c)$ is a Cantor set and the restriction of Q_c to $J(Q_c)$ is conjugate to the one-sided shift on two letters.



$Q_c, c < -2. J(Q_c) = K(Q_c)$ is a Cantor set.