MATH 614

Dynamical Systems and Chaos

Lecture 37:
Ergodic theorems.
Ergodicity.

Measure-preserving transformation

Definition. A **measured space** is a triple $(X, \mathcal{B}, \mu)$, where X is a set, $\mathcal{B}$ is a σ-algebra of (measurable) subsets of X, and $\mu : \mathcal{B} \to [0, \infty]$ is a σ-additive measure on X (finite or σ-finite).

A mapping $T : X \to X$ is called **measurable** if preimage of any measurable set under T is also measurable: $E \in \mathcal{B} \implies T^{-1}(E) \in \mathcal{B}$.

A measurable mapping $T : X \to X$ is called **measure-preserving** if for any $E \in \mathcal{B}$ one has $\mu(T^{-1}(E)) = \mu(E)$.

Borel sets

Proposition Given a collection S of subsets of X, there exists a minimal σ-algebra of subsets of X that contains S.

Suppose X is a topological space. The **Borel** σ**-algebra** $\mathcal{B}(X)$ is the minimal σ-algebra that contains all open subsets of X. Elements of $\mathcal{B}(X)$ are called **Borel sets**.

A mapping $F : X \to X$ is measurable relative to $\mathcal{B}(X)$ if and only if the preimage of any open set is Borel. In particular, each continuous map is measurable.

Recurrence

$(X, \mathcal{B}, \mu)$: measured space
$T : X \to X$: measure-preserving mapping

Let E be a measurable subset of X. A point $x \in E$ is called **recurrent** if $T^n(x) \in E$ for some $n \geq 1$.

A point $x \in E$ is called **infinitely recurrent** if the orbit $x, T(x), T^2(x), \ldots$ visits E infinitely many times.

Theorem (Poincaré 1890) Suppose μ is a finite measure. Then almost all points of E are infinitely recurrent.

Lemma 1 Suppose μ is a finite measure and $\mu(E) > 0$. Then there exists a recurrent point $x \in E$.

Proof: Let $E_0 = E$, $E_1 = T^{-1}(E)$, $E_2 = T^{-1}(E_1) = T^{-2}(E)$, $\dots$, $E_n = T^{-1}(E_{n-1}) = T^{-n}(E)$, $\dots$ Suppose $E_n \cap E_m \neq \emptyset$ for some n and m, $0 \leq n < m$. Take any point $x \in E_n \cap E_m$ and let $y = T^n(x)$. Since $T^n(x), T^m(x) \in E$, it follows that $y \in E$ and $T^{m-n}(y) \in E$, hence y is a recurrent point.

Now assume that sets $E_0, E_1, E_2, \dots$ are disjoint.
Since T preserves measure, we have $\mu(E_{n+1}) = \mu(E_n)$ for all $n \geq 0$ so that $\mu(E_n) = \mu(E) > 0$ for all n.
Then $\mu(E_0 \cup E_1 \cup E_2 \cup \dots) = \infty$, a contradiction.

Lemma 2 Suppose μ is a finite measure. Then almost all points of E are recurrent.

Proof: Let E_∞ denote the set of all non-recurrent points of E. This set is measurable: $E_\infty = E \setminus \big(T^{-1}(E) \cup T^{-2}(E) \cup \dots\big)$. Clearly, no points of E_∞ are recurrent (relative to E_∞). By Lemma 1, $\mu(E_\infty) = 0$.

Individual ergodic theorem

Let $(X, \mathcal{B}, \mu)$ be a measured space and $T : X \to X$ be a measure-preserving transformation.

Birkhoff's Ergodic Theorem For any function $f \in L_1(X, \mu)$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = f^*(x)$$

exists for almost all $x \in X$. The function f^* is T-invariant, i.e., $f^* \circ T = f^*$ almost everywhere. If μ is finite then $f^* \in L_1(X, \mu)$ and

$$\int_X f^* \, d\mu = \int_X f \, d\mu.$$

Ergodicity

Let $(X, \mathcal{B}, \mu)$ be a measured space and $T : X \to X$ be a measure-preserving transformation.

We say that a measurable set $E \subset X$ is **invariant** under T if $\mu(E \triangle T^{-1}(E)) = 0$, that is, if $E = T^{-1}(E)$ up to a set of zero measure. In particular, if $T(E) \subset E$ then $E \subset T^{-1}(E)$ so that $\mu(E \triangle T^{-1}(E)) = \mu(T^{-1}(E) \setminus E) = 0$.

Note that there is a measurable set $E_0 \subset E$ such that $\mu(E \triangle E_0) = 0$ and $T^{-1}(E_0) = E_0$. Namely, let $E_1 = E \cup T^{-1}(E) \cup T^{-2}(E) \cup \dots$. Then $E \subset E_1$, $\mu(E_1 \setminus E) = 0$, $\mu(E_1 \triangle T^{-1}(E_1)) = 0$, and $T^{-1}(E_1) \subset E_1$. Now $E_0 = E_1 \cap T^{-1}(E_1) \cap T^{-2}(E_1) \cap \dots$

Definition. The transformation T is called **ergodic** with respect to μ if any T-invariant measurable set E has either zero or full measure: $\mu(E) = 0$ or $\mu(X \setminus E) = 0$.

Birkhoff's Ergodic Theorem (ergodic case)

Suppose μ is finite and T is ergodic. Given $f \in L_1(X, \mu)$, for almost all $x \in X$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = \frac{1}{\mu(X)} \int_X f \, d\mu.$$

(time average is equal to space average)

In the case $f = \chi_E$ $(E \in \mathcal{B})$, we obtain

$$\lim_{n\to\infty} \frac{\#\{0 \le k \le n-1 \mid T^k(x) \in E\}}{n} = \frac{\mu(E)}{\mu(X)}.$$

(almost every orbit is uniformly distributed)

Koopman's operator

$(X, \mathcal{B}, \mu)$: measured space
$T : X \to X$: measure-preserving transformation

To any function $f : X \to \mathbb{C}$ we assign another function Uf defined by $(Uf)(x) = f(T(x))$ for all $x \in X$.

Linear functional operator U: $f \mapsto Uf$.

Proposition If f is integrable then so is Uf. Moreover,

$$\int_X Uf \, d\mu = \int_X f(T(x)) \, d\mu(x) = \int_X f \, d\mu.$$

$f \in L_2(X, \mu)$ means that $\int_X |f|^2 \, d\mu < \infty$.

$L_2(X, \mu)$ is a Hilbert space with respect to the inner product

$$(f, g) = \int_X f(x)\overline{g(x)} \, d\mu(x).$$

Let T be a measure-preserving transformation and U be the associated operator, $Uf = f \circ T$.

Then $U(L_2(x, \mu)) \subset L_2(X, \mu)$. Furthermore,

$$(Uf, Ug) = (f, g)$$

for all $f, g \in L_2(X, \mu)$.

That is, U is an **isometric** operator on the Hilbert space $L_2(X, \mu)$. If T is invertible and T^{-1} is also measure-preserving, then U is a **unitary** operator.

Mean ergodic theorem

von Neumann's Ergodic Theorem Suppose U is an isometric operator in a Hilbert space $\mathcal{H}$. Then for any $f \in \mathcal{H}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f = f^* \text{ (in } \mathcal{H}\text{)},$$

where $f^* \in \mathcal{H}$ is the orthogonal projection of f on the subspace of U-invariant functions in $\mathcal{H}$.

Namely, $Uf^* = f^*$ and $(f - f^*, g) = 0$ for any element $g \in \mathcal{H}$ such that $Ug = g$.

If U is associated to a measure-preserving map $T : X \to X$, then for any $f \in L_2(X, \mu)$ we have

$$\lim_{n\to\infty} \int_X \left| \frac{1}{n} \sum_{k=0}^{n-1} U^k f - f^* \right|^2 d\mu \to 0,$$

where $f^* \in L_2(X, \mu)$ and $Uf^* = f^*$.

Lemma T is ergodic if and only if $Uf = f$ for a measurable function f implies f is constant (almost everywhere).

If T is ergodic then

$$\lim_{n\to\infty} \int_X \left| \frac{1}{n} \sum_{k=0}^{n-1} U^k f - c \right|^2 d\mu \to 0,$$

where

$$c = \frac{1}{\mu(X)} \int_X f \, d\mu.$$

Rotations of the circle

Measured space $(S^1, \mathcal{B}(S^1), \mu)$, where μ is the length measure on S^1.

R_α: rotation of the unit circle by angle α.
R_α is a measure-preserving homeomorphism.

Theorem If α is not commensurable with π, then the rotation R_α is ergodic.

Let U_α be the associated operator on $L_2(S^1, \mu)$.

Relative to the angular coordinate on S^1, elements of $L_2(S^1, \mu)$ are 2π-periodic functions on $\mathbb{R}$. The inner product is given by

$$(f, g) = \int_0^{2\pi} f(x)\overline{g(x)}\, dx.$$

The operator U_α acts as follows:

$$(U_\alpha f)(x) = f(x + \alpha), \quad x \in \mathbb{R}.$$

For any $m \in \mathbb{Z}$ let $h_m(x) = e^{imx}$, $x \in \mathbb{R}$. Then $h_m \in L_2(S^1, \mu)$ and $U_\alpha h_m = e^{im\alpha} h_m$ so that h_m is an eigenfunction of U_α. Note that $\{h_m\}_{m \in \mathbb{Z}}$ is an orthogonal basis of the Hilbert space $L_2(X, \mu)$. We say that U_α has **pure point spectrum**.

Any $f \in L_2(X, \mu)$ is uniquely expanded as

$$f = \sum_{m \in \mathbb{Z}} c_m h_m, \quad \text{(Fourier series)}$$

where $c_m \in \mathbb{C}$. Then

$$U_\alpha f = \sum_{m \in \mathbb{Z}} e^{im\alpha} c_m h_m.$$

Hence $U_\alpha f = f$ only if $(e^{im\alpha} - 1)c_m = 0$ for all $m \in \mathbb{Z}$. That is, if $f = c_0$, a constant.