

Math 304-504

Linear algebra

**Lecture 40:**  
**Review for the final exam.**

## Topics for the final exam: Part I

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.
- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.
- Matrix algebra. Inverse matrix.
- Determinants: explicit formulas for  $2 \times 2$  and  $3 \times 3$  matrices, row and column expansions, elementary row and column operations.

## Topics for the final exam: Part II

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension. Rank and nullity of a matrix.
- Change of coordinates, transition matrix.
- Linear mappings/transformations/operators.  
Range and kernel of a linear mapping.
- Matrix of a linear mapping. Similar matrices.

## Topics for the final exam: Part III

- Norms. Inner products.
- Orthogonality. Orthogonal complement.
- Least squares problems.
- Orthogonal and orthonormal bases. The Gram-Schmidt orthogonalization process.
- Eigenvalues, eigenvectors, and eigenspaces. Characteristic polynomial.
- Bases of eigenvectors. Diagonalization.

## Topics for the final exam: Part IV

- Matrix exponentials.
- Orthonormal bases of eigenvectors. Symmetric and normal matrices.
- Orthogonal matrices. Rotations in space.
- Orthogonal polynomials.
- Markov chains.

## Bases of eigenvectors

Let  $A$  be an  $n \times n$  matrix with real entries.

- $A$  has  $n$  distinct real eigenvalues  $\implies$  a basis for  $\mathbb{R}^n$  formed by eigenvectors of  $A$
- $A$  has complex eigenvalues  $\implies$  no basis for  $\mathbb{R}^n$  formed by eigenvectors of  $A$
- $A$  has  $n$  distinct complex eigenvalues  $\implies$  a basis for  $\mathbb{C}^n$  formed by eigenvectors of  $A$
- $A$  has multiple eigenvalues  $\implies$  further information is needed
- an orthonormal basis for  $\mathbb{R}^n$  formed by eigenvectors of  $A$   
 $\iff A$  is symmetric:  $A^T = A$
- an orthonormal basis for  $\mathbb{C}^n$  formed by eigenvectors of  $A$   
 $\iff A$  is normal:  $AA^T = A^T A$

**Problem** For each of the following matrices determine whether it allows

(a) a basis of eigenvectors for  $\mathbb{R}^n$ ,

(b) a basis of eigenvectors for  $\mathbb{C}^n$ ,

(c) an orthonormal basis of eigenvectors for  $\mathbb{R}^n$ ,

(d) an orthonormal basis of eigenvectors for  $\mathbb{C}^n$ .

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad \text{(a),(b),(c),(d): yes}$$

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{(a),(b),(c),(d): no}$$

**Problem** For each of the following matrices determine whether it allows

(a) a basis of eigenvectors for  $\mathbb{R}^n$ ,

(b) a basis of eigenvectors for  $\mathbb{C}^n$ ,

(c) an orthonormal basis of eigenvectors for  $\mathbb{R}^n$ ,

(d) an orthonormal basis of eigenvectors for  $\mathbb{C}^n$ .

$$C = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \quad \text{(a),(b): yes} \quad \text{(c),(d): no}$$

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{(b),(d): yes} \quad \text{(a),(c): no}$$

**Problem** For each of the following matrices determine whether it allows

(a) a basis of eigenvectors for  $\mathbb{R}^n$ ,

(b) a basis of eigenvectors for  $\mathbb{C}^n$ ,

(c) an orthonormal basis of eigenvectors for  $\mathbb{R}^n$ ,

(d) an orthonormal basis of eigenvectors for  $\mathbb{C}^n$ .

(a),(b),(d): yes      (c): no      *Impossible!*

(b): yes      (a),(c),(d): no       $E = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$

**Problem** Let  $V$  be the vector space spanned by functions  $f_1(x) = x \sin x$ ,  $f_2(x) = x \cos x$ ,  $f_3(x) = \sin x$ , and  $f_4(x) = \cos x$ . Consider the linear operator  $D : V \rightarrow V$ ,  $D = d/dx$ .

- (a) Find the matrix  $A$  of the operator  $D$  relative to the basis  $f_1, f_2, f_3, f_4$ .
- (b) Find the eigenvalues of  $A$ .
- (c) Is the matrix  $A$  diagonalizable in  $\mathbb{R}^4$  (in  $\mathbb{C}^4$ )?

$A$  is a  $4 \times 4$  matrix whose columns are coordinates of functions  $Df_i = f_i'$  relative to the basis  $f_1, f_2, f_3, f_4$ .

$$f_1'(x) = (x \sin x)' = x \cos x + \sin x = f_2(x) + f_3(x),$$

$$\begin{aligned} f_2'(x) &= (x \cos x)' = -x \sin x + \cos x \\ &= -f_1(x) + f_4(x), \end{aligned}$$

$$f_3'(x) = (\sin x)' = \cos x = f_4(x),$$

$$f_4'(x) = (\cos x)' = -\sin x = -f_3(x).$$

Thus  $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$

Eigenvalues of  $A$  are roots of its characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & -1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\begin{aligned} \det(A - \lambda I) &= -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} - (-1) \begin{vmatrix} 1 & 0 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} \\ &= \lambda^2(\lambda^2 + 1) + (\lambda^2 + 1) = (\lambda^2 + 1)^2. \end{aligned}$$

The eigenvalues are  $i$  and  $-i$ , both of multiplicity 2.

Complex eigenvalues  $\implies A$  is not diagonalizable in  $\mathbb{R}^4$

If  $A$  is diagonalizable in  $\mathbb{C}^4$  then  $A = UXU^{-1}$ , where  $U$  is an invertible matrix with complex entries and

$$X = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

This would imply that  $A^2 = UX^2U^{-1}$ . But  $X^2 = -I$  so that  $A^2 = U(-I)U^{-1} = -I$ .

$$A^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}.$$

Since  $A^2 \neq -I$ , the matrix  $A$  is not diagonalizable in  $\mathbb{C}^4$ .

**Problem** Consider a linear operator  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by  $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}$ , where  $\mathbf{v}_0 = (3/5, 0, -4/5)$ .

- (a) Find the matrix  $B$  of the operator  $L$ .
- (b) Find the range and kernel of  $L$ .
- (c) Find the eigenvalues of  $L$ .
- (d) Find the matrix of the operator  $L^{2007}$  ( $L$  applied 2007 times).

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}, \quad \mathbf{v}_0 = (3/5, 0, -4/5).$$

Let  $\mathbf{v} = (x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ . Then

$$\begin{aligned} L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 3/5 & 0 & -4/5 \\ x & y & z \end{vmatrix} \\ &= \frac{4}{5}y\mathbf{e}_1 - \left(\frac{4}{5}x + \frac{3}{5}z\right)\mathbf{e}_2 + \frac{3}{5}y\mathbf{e}_3. \end{aligned}$$

In particular,  $L(\mathbf{e}_1) = -\frac{4}{5}\mathbf{e}_2$ ,  $L(\mathbf{e}_2) = \frac{4}{5}\mathbf{e}_1 + \frac{3}{5}\mathbf{e}_3$ ,  
 $L(\mathbf{e}_3) = -\frac{3}{5}\mathbf{e}_2$ .

Therefore  $B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}$ .

$$B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}.$$

The range of the operator  $L$  coincides with the column space of the matrix  $B$ . It follows that  $\text{Range}(L)$  is the plane spanned by vectors  $\mathbf{v}_1 = (0, 1, 0)$  and  $\mathbf{v}_2 = (4, 0, 3)$ .

The kernel of  $L$  is the solution set for the equation  $B\mathbf{x} = \mathbf{0}$ .

$$\begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies x + \frac{3}{4}z = y = 0 \implies \mathbf{x} = t(-3/4, 0, 1).$$

Alternatively, the kernel of  $L$  is the set of vectors  $\mathbf{v} \in \mathbb{R}^3$  such that  $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \mathbf{0}$ .

It follows that  $\ker(L)$  is the line spanned by  $\mathbf{v}_0 = (3/5, 0, -4/5)$ .

Characteristic polynomial of the matrix  $B$ :

$$\begin{aligned} \det(B - \lambda I) &= \begin{vmatrix} -\lambda & 4/5 & 0 \\ -4/5 & -\lambda & -3/5 \\ 0 & 3/5 & -\lambda \end{vmatrix} \\ &= -\lambda^3 - (3/5)^2\lambda - (4/5)^2\lambda = -\lambda^3 - \lambda = -\lambda(\lambda^2 + 1). \end{aligned}$$

The eigenvalues are  $0$ ,  $i$ , and  $-i$ .

The matrix of the operator  $L^{2007}$  is  $B^{2007}$ .

Since the matrix  $B$  has eigenvalues  $0$ ,  $i$ , and  $-i$ , it is diagonalizable in  $\mathbb{C}^3$ . Namely,  $B = UDU^{-1}$ , where  $U$  is an invertible matrix with complex entries and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Then  $B^{2007} = UD^{2007}U^{-1}$ . We have that  $D^{2007} = \text{diag}(0, i^{2007}, (-i)^{2007}) = \text{diag}(0, -i, i) = -D$ .

Hence

$$B^{2007} = U(-D)U^{-1} = -B = \begin{pmatrix} 0 & -4/5 & 0 \\ 4/5 & 0 & 3/5 \\ 0 & -3/5 & 0 \end{pmatrix}.$$