

Sample problems for Test 1: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 (20 pts.) Find the point of intersection of the planes $x + 2y - z = 1$, $x - 3y = -5$, and $2x + y + z = 0$ in \mathbb{R}^3 .

The intersection point (x, y, z) is a solution of the system

$$\begin{cases} x + 2y - z = 1, \\ x - 3y = -5, \\ 2x + y + z = 0. \end{cases}$$

To solve the system, we convert its augmented matrix into reduced row echelon form using elementary row operations:

$$\begin{aligned} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & -3 & 0 & -5 \\ 2 & 1 & 1 & 0 \end{array} \right) &\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 1 & -6 \\ 2 & 1 & 1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 1 & -6 \\ 0 & -3 & 3 & -2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -2 \\ 0 & -5 & 1 & -6 \end{array} \right) \\ &\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & -5 & 1 & -6 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & -4 & -\frac{8}{3} \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array} \right) \\ &\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array} \right). \end{aligned}$$

Thus the three planes intersect at the point $(-1, \frac{4}{3}, \frac{2}{3})$.

Alternative solution: The intersection point (x, y, z) is a solution of the system

$$\begin{cases} x + 2y - z = 1, \\ x - 3y = -5, \\ 2x + y + z = 0. \end{cases}$$

Adding all three equations, we obtain $4x = -4$. Hence $x = -1$. Substituting $x = -1$ into the second equation, we obtain $y = \frac{4}{3}$. Substituting $x = -1$ and $y = \frac{4}{3}$ into the third equation, we obtain $z = \frac{2}{3}$. It is easy to check that $x = -1$, $y = \frac{4}{3}$, $z = \frac{2}{3}$ is indeed a solution of the system. Thus $(-1, \frac{4}{3}, \frac{2}{3})$ is the unique intersection point.

Problem 2 (30 pts.) Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A .

First let us subtract 2 times the fourth column of A from the first column:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

Now the determinant can be easily expanded by the fourth row:

$$\begin{vmatrix} -1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 4 \\ 2 & 3 & 2 \\ 0 & 0 & -1 \end{vmatrix}.$$

The 3×3 determinant is easily expanded by the third row:

$$\begin{vmatrix} -1 & -2 & 4 \\ 2 & 3 & 2 \\ 0 & 0 & -1 \end{vmatrix} = (-1) \begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix}.$$

Thus

$$\det A = - \begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix} = -1.$$

Another way to evaluate $\det A$ is to reduce the matrix A to the identity matrix using elementary row operations (see below). This requires much more work but we are going to do it anyway, to find the inverse of A .

(ii) Find the inverse matrix A^{-1} .

First we merge the matrix A with the identity matrix into one 4×8 matrix

$$(A|I) = \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract 2 times the first row from the second row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right).$$

Subtract 2 times the first row from the third row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right).$$

Subtract 2 times the first row from the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array} \right).$$

Multiply the second, the third, and the fourth rows by -1 :

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array} \right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array} \right).$$

Finally the left part of our 4×8 matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of A . Thus

$$A^{-1} = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 16 & -19 \\ -2 & -1 & -10 & 12 \\ 0 & 0 & -1 & 1 \\ -6 & -4 & -32 & 39 \end{pmatrix}.$$

As a byproduct, we can evaluate the determinant of A . We have transformed A into the identity matrix using elementary row operations. These included no row exchanges and three row multiplications, each time by -1 . It follows that $\det I = (-1)^3 \det A$. Hence $\det A = -\det I = -1$.

Problem 3 (20 pts.) Let \mathcal{P}_4 be the vector space of all polynomials (with real coefficients) of degree less than 4. Determine which of the following subsets of \mathcal{P}_4 are vector subspaces. Briefly explain.

(i) The set S_1 of polynomials $p(x) \in \mathcal{P}_4$ such that $p(0) = 0$.

The set S_1 is not empty because it contains the zero polynomial. S_1 is a subspace of \mathcal{P}_4 since it is closed under addition and scalar multiplication. Alternatively, S_1 is the kernel of a linear functional $\ell : \mathcal{P}_4 \rightarrow \mathbb{R}$ given by $\ell(p) = p(0)$.

(ii) The set S_2 of polynomials $p(x) \in \mathcal{P}_4$ such that $p(0)p(1) = 0$.

A polynomial $p(x) \in \mathcal{P}_4$ belongs to S_2 if $p(0) = 0$ or $p(1) = 0$. The set S_2 is not a subspace because it is not closed under addition. For example, the polynomials $p_1(x) = x$ and $p_2(x) = x - 1$ belong to S_2 while their sum $p(x) = 2x - 1$ is not in S_2 .

(iii) The set S_3 of polynomials $p(x) \in \mathcal{P}_4$ such that $(p(0))^2 + (p(1))^2 = 0$.

A polynomial $p(x) \in \mathcal{P}_4$ belongs to S_3 if $p(0) = p(1) = 0$. The set S_3 is not empty because it contains the zero polynomial. S_3 is a subspace of \mathcal{P}_4 since it is closed under addition and scalar multiplication. Alternatively, S_3 is the kernel of a linear mapping $L : \mathcal{P}_4 \rightarrow \mathbb{R}^2$ given by $L(p) = (p(0), p(1))$.

(iv) The set S_4 of polynomials $p(x) \in \mathcal{P}_4$ such that $p(0) = 0$ and $p(1) = 1$.

The set S_4 is not a subspace of \mathcal{P}_4 because it does not contain the zero polynomial (consequently, S_4 is not closed under scalar multiplication).

Problem 4 (30 pts.) Let $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

(i) Find the rank and the nullity of the matrix B .

The rank (dimension of the row space) and the nullity (dimension of the nullspace) of a matrix are preserved under elementary row operations. We apply such operations to convert the matrix B into row echelon form.

First interchange the first row with the second row:

$$\begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}.$$

Add 3 times the first row to the third row, then subtract 2 times the first row from the fourth row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 2 & -1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}.$$

Multiply the second row by -1 :

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}.$$

Add the fourth row to the third row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 3 \end{pmatrix}.$$

Add 3 times the second row to the fourth row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -16 & 0 \end{pmatrix}.$$

Add 16 times the third row to the fourth row:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -16 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now that the matrix is in row echelon form, its rank equals the number of nonzero rows, which is 3. Since

$$(\text{rank of } B) + (\text{nullity of } B) = (\text{the number of columns of } B) = 4,$$

it follows that the nullity of B equals 1.

(ii) Find a basis for the row space of B , then extend this basis to a basis for \mathbb{R}^4 .

The row space of a matrix is invariant under elementary row operations. Therefore the row space of the matrix B is the same as the row space of its row echelon form

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The nonzero rows of the latter are linearly independent so that they form a basis for its row space. Hence the vectors $\mathbf{v}_1 = (1, 1, 2, -1)$, $\mathbf{v}_2 = (0, 1, -4, -1)$, and $\mathbf{v}_3 = (0, 0, 1, 0)$ form a basis for the row space of B .

To extend the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to a basis for \mathbb{R}^4 , we need a vector $\mathbf{v}_4 \in \mathbb{R}^4$ that is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. It is known that at least one of the vectors $\mathbf{e}_1 = (1, 0, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, 0)$, $\mathbf{e}_3 = (0, 0, 1, 0)$, and $\mathbf{e}_4 = (0, 0, 0, 1)$ can be chosen as \mathbf{v}_4 . In particular, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_4$ form a basis for \mathbb{R}^4 . This follows from the fact that the 4×4 matrix whose rows are these vectors is not singular:

$$\begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Bonus Problem 5 (25 pts.) Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Suppose that $af_1(x) + bf_2(x) + cf_3(x) = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Differentiating the identity $af_1(x) + bf_2(x) + cf_3(x) = 0$ four times, we obtain four more identities:

$$\begin{aligned} ax + bxe^x + ce^{-x} &= 0, \\ a + be^x + bxe^x - ce^{-x} &= 0, \\ 2be^x + bxe^x + ce^{-x} &= 0, \\ 3be^x + bxe^x - ce^{-x} &= 0, \\ 4be^x + bxe^x + ce^{-x} &= 0. \end{aligned}$$

Subtracting the third identity from the fifth one, we obtain $2be^x = 0$, which implies that $b = 0$. Substituting $b = 0$ in the third identity, we obtain $ce^{-x} = 0$, which implies that $c = 0$. Substituting $b = 0$ and $c = 0$ in the second identity, we obtain $a = 0$.

Alternative solution: Suppose that $ax + bxe^x + ce^{-x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

For any $x \neq 0$ divide both sides of the identity by xe^x :

$$ae^{-x} + b + cx^{-1}e^{-2x} = 0.$$

Note that $e^{-x} \rightarrow 0$ and $x^{-1}e^{-2x} \rightarrow 0$ as $x \rightarrow +\infty$. Hence the left-hand side approaches b as $x \rightarrow +\infty$. It follows that $b = 0$. Now $ax + ce^{-x} = 0$ for all $x \in \mathbb{R}$. For any $x \neq 0$ divide both sides of the latter identity by x :

$$a + cx^{-1}e^{-x} = 0.$$

Since $x^{-1}e^{-x} \rightarrow 0$ as $x \rightarrow +\infty$, the left-hand side approaches a as $x \rightarrow +\infty$. It follows that $a = 0$. Then $ce^{-x} = 0$, which implies that $c = 0$.

Bonus Problem 6 (20 pts.) Let V and W be subspaces of the vector space \mathbb{R}^n such that $V \cup W$ is also a subspace of \mathbb{R}^n . Show that $V \subset W$ or $W \subset V$.

Assume the contrary: neither of the subspaces V and W is contained in the other. Then there exist vectors $\mathbf{v} \in V$ and $\mathbf{w} \in W$ such that $\mathbf{v} \notin W$ and $\mathbf{w} \notin V$. Let $\mathbf{x} = \mathbf{v} + \mathbf{w}$. Since $\mathbf{v}, \mathbf{w} \in V \cup W$ and $V \cup W$ is a subspace, it follows that $\mathbf{x} \in V \cup W$. That is, $\mathbf{x} \in V$ or $\mathbf{x} \in W$. However in the case $\mathbf{x} \in V$ we have $\mathbf{w} = \mathbf{x} - \mathbf{v} \in V$, while in the case $\mathbf{x} \in W$ we have $\mathbf{v} = \mathbf{x} - \mathbf{w} \in W$. In either case we arrive at a contradiction. Thus the initial assumption was wrong. That is, one of the subspaces V and W does contain the other.