Final exam

Problem 1 (25 pts.) Find a quadratic polynomial $p(x)$ such that $p(1) = 2$, $p(2) = 3$, and $p(3) = p(-1)$.

Problem 2 (30 pts.) Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0, \quad \text{where } \mathbf{v}_0 = (1, 1, -1).$$

(i) Find the matrix of the operator L.
(ii) Find the dimensions of the image and the null-space of L.
(iii) Find bases for the image and the null-space of L.

Problem 3 (30 pts.) Let $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.
(ii) Find the inverse matrix A^{-1}.

Problem 4 (35 pts.) Let $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.
(ii) For each eigenvalue of B, find an associated eigenvector.
(iii) Find a diagonal matrix Λ and an invertible matrix U such that $B = U\Lambda U^{-1}$.

Problem 5 (30 pts.) Let V be a three-dimensional subspace of $\mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1, 1, 0, 0)$, $\mathbf{x}_2 = (1, 3, 1, 1)$, and $\mathbf{x}_3 = (1, 1, -3, -1)$.

(i) Find an orthogonal basis for V.
(ii) Find the distance from the point $\mathbf{y} = (2, 0, 2, 4)$ to the subspace V.

Bonus Problem 6 (25 pts.) Let ℓ_1 be the line passing through the points $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (1, 2, 3)$. Let ℓ_2 be the line passing through the points $\mathbf{c} = (1, -1, 0)$ and $\mathbf{d} = (2, -2, 1)$. Let Π be the plane that contains the line ℓ_1 and is parallel to the line ℓ_2.

(i) Find a parametric representation for the plane Π.
(ii) Find an equation for the plane Π.
(iii) Find the distance from the line ℓ_2 to the plane Π.

Bonus Problem 7 (30 pts.) Let $C = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$. Find the matrix C^9.