Math 311-503 May 9, 2007

Final exam: Solutions

Problem 1 (25 pts.) Find a quadratic polynomial $p(x)$ such that $p(1) = 2$, $p(2) = 3$, and $p(3) = p(-1)$.

Solution: $p(x) = x^2 - 2x + 3$.

A quadratic polynomial $p(x) = ax^2 + bx + c$ is the desired one if its coefficients satisfy the following relations:

$$\begin{cases} a + b + c = 2, \\ 4a + 2b + c = 3, \\ 9a + 3b + c = a - b + c. \end{cases}$$

Solving this system of linear equations, we obtain that

$$\begin{cases} a + b + c = 2 \\ 4a + 2b + c = 3 \\ 9a + 3b + c = a - b + c \end{cases} \iff \begin{cases} a + b + c = 2 \\ 4a + 2b + c = 3 \\ 8a + 4b = 0 \end{cases} \iff \begin{cases} a + b + c = 2 \\ 3a + b = 1 \\ 2a + b = 0 \end{cases}$$

$$\iff \begin{cases} a + b + c = 2 \\ a = 1 \\ 2a + b = 0 \end{cases} \iff \begin{cases} a + b + c = 2 \\ a = 1 \\ b = -2 \end{cases} \iff \begin{cases} a = 1 \\ b = -2 \\ c = 3 \end{cases}$$

Thus $p(x) = x^2 - 2x + 3$.

Problem 2 (30 pts.) Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0, \quad \text{where } \mathbf{v}_0 = (1, 1, -1).$$

(i) Find the matrix of the operator L.

Solution: $\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

Given $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, we have that

$$L(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ 1 & 1 & -1 \end{vmatrix} = (-y - z)\mathbf{i} + (x + z)\mathbf{j} + (x - y)\mathbf{k},$$

where $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ is the standard basis for $\mathbb{R}^3$. Let A denote the matrix of the linear operator L. Then $L(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$. It follows that vectors $L(\mathbf{i}), L(\mathbf{j}), L(\mathbf{k})$ are columns of A. By the above $L(\mathbf{i}) = \mathbf{j} + \mathbf{k}$, $L(\mathbf{j}) = -\mathbf{i} - \mathbf{k}$, and $L(\mathbf{k}) = -\mathbf{i} + \mathbf{j}$. Therefore

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

(ii) Find the dimensions of the image and the null-space of L.

Solution: $\dim \operatorname{Im} L = 2$, $\dim \operatorname{Null} L = 1$.

The null-space $\operatorname{Null} L$ of the operator L is the subspace of all vectors $\mathbf{v} \in \mathbb{R}^3$ such that $L(\mathbf{v}) = \mathbf{0}$. Note that $\mathbf{v} \times \mathbf{v}_0 = \mathbf{0}$ if and only if the vectors $\mathbf{v}$ and $\mathbf{v}_0$ are parallel. Therefore $\operatorname{Null} L$ is the line spanned by $\mathbf{v}_0$. Its dimension is 1.

For any linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ we have that

$$\dim \operatorname{Im} L + \dim \operatorname{Null} L = \dim \mathbb{R}^3 = 3.$$

Since $\dim \operatorname{Null} L = 1$, it follows that $\dim \operatorname{Im} L = 2$.

(iii) Find bases for the image and the null-space of L.

Solution: $(0, 1, 1)$, $(-1, 0, -1)$ is a basis for $\operatorname{Im} L$; $(1, 1, -1)$ is a basis for $\operatorname{Null} L$.

Since the null-space of L is the line spanned by the vector $\mathbf{v}_0 = (1, 1, -1)$, this vector is a basis for the null-space.

The image $\operatorname{Im} L$ of the linear operator L is the subspace of all vectors of the form $L(\mathbf{v})$, where $\mathbf{v} \in \mathbb{R}^3$. It is spanned by columns of the matrix of L, that is, by vectors $\mathbf{v}_1 = (0, 1, 1)$, $\mathbf{v}_2 = (-1, 0, -1)$, and $\mathbf{v}_3 = (-1, 1, 0)$. It is easy to see that $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$. Hence the image is spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ alone. Clearly, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent vectors. Therefore $\mathbf{v}_1, \mathbf{v}_2$ is a basis for $\operatorname{Im} L$.

Problem 3 (30 pts.) Let $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.

Solution: $\det A = 1$.

The determinant of A is easily evaluated using row or column expansions. For example, let us expand the determinant by the first row:

$$\begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{vmatrix} = -(-1) \cdot \begin{vmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}.$$

Then expand the 3-by-3 determinant by the first column:

$$\det A = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1.$$

Another way to evaluate $\det A$ is to reduce the matrix A to the identity matrix using elementary row operations (see below). This requires more work but we are going to do it anyway, to find the inverse of A.

(ii) Find the inverse matrix A^{-1}.

Solution: $A^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

First we merge the matrix A with the identity matrix into one 4-by-8 matrix:

$$\left(\begin{array}{rrrr|rrrr} 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Interchange the first row with the second row:

$$\left(\begin{array}{rrrr|rrrr} 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Add the second row to the fourth row:

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

Interchange the third row with the fourth row:

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right).$$

Multiply the second and the third rows by -1:

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right).$$

Add the fourth row to the first row:

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right).$$

Subtract the third row from the first row:

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right).$$

Finally the left part of our 4-by-8 matrix is transformed into the identity matrix. Therefore the current right side is the inverse matrix of A. Thus

$$A^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

As a byproduct, we can evaluate the determinant of A. We have transformed A into the identity matrix using elementary row operations. These included two row exchanges and two row multiplications by -1. It follows that $\det A = (-1)^2 \det I = 1$.

Problem 4 (35 pts.) Let $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.

Solution: -1, 0, 1.

The eigenvalues of B are roots of the characteristic equation $\det(B - \lambda I) = 0$. One obtains that

$$\det(B-\lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = (-\lambda)^3 + \lambda = \lambda(1-\lambda)(1+\lambda).$$

Hence the matrix B has three eigenvalues: 0, 1, and -1.

(ii) For each eigenvalue of B, find an associated eigenvector.

Solution: $(-1,1,0)$, $(-1,0,1)$, and $(1,1,0)$ are eigenvectors of B associated with the eigenvalues -1, 0, and 1, respectively.

An eigenvector $\mathbf{x} = (x,y,z)$ of B associated with an eigenvalue λ is a nonzero solution of the vector equation $(B - \lambda I)\mathbf{x} = \mathbf{0}$. First consider the case $\lambda = 0$. We obtain that

$$B\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x + z = 0, \\ y = 0. \end{cases}$$

The general solution is $x = -t$, $y = 0$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_0 = (-1,0,1)$ is an eigenvector of B associated with the eigenvalue 0.

Next consider the case $\lambda = 1$. We obtain that

$$(B - I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - y = 0, \\ z = 0. \end{cases}$$

The general solution is $x = t$, $y = t$, $z = 0$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (1,1,0)$ is an eigenvector of B associated with the eigenvalue 1.

Finally, consider the case $\lambda = -1$. We obtain that

$$(B + I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x + y = 0, \\ z = 0. \end{cases}$$

The general solution is $x = -t$, $y = t$, $z = 0$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_{-1} = (-1, 1, 0)$ is an eigenvector of B associated with the eigenvalue -1.

(iii) Find a diagonal matrix Λ and an invertible matrix U such that $B = U\Lambda U^{-1}$.

Solution: $\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

The vectors $\mathbf{v}_{-1}$, $\mathbf{v}_0$, and $\mathbf{v}_1$ are linearly independent since they are eigenvectors of the matrix B associated with distinct eigenvalues. Therefore $\mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1$ is a basis for $\mathbb{R}^3$. Let U be the matrix whose columns are vectors $\mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1$. Then U is invertible and we have $B = U\Lambda U^{-1}$, where Λ is the matrix of the linear operator $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(\mathbf{v}) = B\mathbf{v}$ relative to the basis $\mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1$. Since $B\mathbf{v}_{-1} = -\mathbf{v}_{-1}$, $B\mathbf{v}_0 = \mathbf{0}$, and $B\mathbf{v}_1 = \mathbf{v}_1$, it follows that $\Lambda = \operatorname{diag}(-1, 0, 1)$.

Problem 5 (30 pts.) Let V be a three-dimensional subspace of $\mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1, 1, 0, 0)$, $\mathbf{x}_2 = (1, 3, 1, 1)$, and $\mathbf{x}_3 = (1, 1, -3, -1)$.

(i) Find an orthogonal basis for V.

Solution: $(1, 1, 0, 0), (-1, 1, 1, 1), (-1, 1, -2, 0)$.

To find an orthogonal basis for the subspace V, we apply the Gram-Schmidt orthogonalization process to the spanning set $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 = (1, 1, 0, 0), \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (1, 3, 1, 1) - \frac{4}{2}(1, 1, 0, 0) = (-1, 1, 1, 1), \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = (1, 1, -3, -1) - \frac{2}{2}(1, 1, 0, 0) - \frac{-4}{4}(-1, 1, 1, 1) = (-1, 1, -2, 0).
\end{aligned}$$

By construction, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are orthogonal to each other. It follows that they are linearly independent. Also, the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is the same as the span of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, that is, the subspace V. Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthogonal basis for V.

(ii) Find the distance from the point $\mathbf{y} = (2, 0, 2, 4)$ to the subspace V.

Solution: $2\sqrt{3}$.

The distance from $\mathbf{y}$ to V is the distance from $\mathbf{y}$ to the closest point $\mathbf{y}_0$ in V, which is the orthogonal projection of $\mathbf{y}$ on V. Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthogonal basis for V, it follows that

$$\mathbf{y}_0 = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{y} \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3$$

$$= \frac{2}{2}(1, 1, 0, 0) + \frac{4}{4}(-1, 1, 1, 1) + \frac{-6}{6}(-1, 1, -2, 0) = (1, 1, 3, 1).$$

Then $\mathbf{y} - \mathbf{y}_0 = (2, 0, 2, 4) - (1, 1, 3, 1) = (1, -1, -1, 3)$ and the desired distance is $|\mathbf{y} - \mathbf{y}_0| = \sqrt{12} = 2\sqrt{3}$.

Bonus Problem 6 (25 pts.) Let ℓ_1 be the line passing through the points $\mathbf{a} = (1, 1, 1)$ and $\mathbf{b} = (1, 2, 3)$. Let ℓ_2 be the line passing through the points $\mathbf{c} = (1, -1, 0)$ and $\mathbf{d} = (2, -2, 1)$. Let Π be the plane that contains the line ℓ_1 and is parallel to the line ℓ_2.

(i) Find a parametric representation for the plane Π.

Solution: $(1,1,1) + t_1(0,1,2) + t_2(1,-1,1)$.

By definition of the plane Π, it passes through the point $\mathbf{a} = (1,1,1)$ and the vectors $\mathbf{v}_1 = \mathbf{b} - \mathbf{a} = (0,1,2)$ and $\mathbf{v}_2 = \mathbf{d} - \mathbf{c} = (1,-1,1)$ are parallel to Π. Clearly, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Thus we get a parametric representation $\mathbf{a} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 = (1,1,1) + t_1(0,1,2) + t_2(1,-1,1)$ for the plane Π.

(ii) Find an equation for the plane Π.

Solution: $3x + 2y - z = 4$.

Since the vectors $\mathbf{v}_1 = (0,1,2)$ and $\mathbf{v}_2 = (1,-1,1)$ are parallel to the plane Π, the vector $\mathbf{w} = \mathbf{v}_1 \times \mathbf{v}_2$ is orthogonal to Π. We obtain

$$\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k} = (3,2,-1).$$

It follows that the plane Π is given by an equation $3x + 2y - z = c$, where c is a constant. Using the fact that the point $\mathbf{a} = (1,1,1)$ lies on Π, we get $c = 4$.

(iii) Find the distance from the line ℓ_2 to the plane Π.

Solution: $3/\sqrt{14}$.

Since the line ℓ_2 is parallel to the plane Π, all points of ℓ_2 lie at the same distance from Π. In particular, the distance from ℓ_2 to Π is the same as the distance from the point $\mathbf{c} = (1,-1,0)$ to Π. As the plane Π is given by the equation $3x + 2y - z = 4$, the latter distance is equal to

$$\frac{|3\cdot 1 + 2\cdot(-1) - 1\cdot 0 - 4|}{\sqrt{3^2 + 2^2 + (-1)^2}} = \frac{3}{\sqrt{14}}.$$

Bonus Problem 7 (30 pts.) Let $C = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$. Find the matrix C^9.

Solution: $C^9 = \begin{pmatrix} 0 & -81 & -81 \\ 81 & 0 & 81 \\ 81 & -81 & 0 \end{pmatrix}$.

Let q denote the characteristic polynomial of the matrix C. We obtain that

$$q(\lambda) = \det(C - \lambda I) = \begin{vmatrix} -\lambda & -1 & -1 \\ 1 & -\lambda & 1 \\ 1 & -1 & -\lambda \end{vmatrix} = -\lambda^3 - 3\lambda.$$

By the Cayley-Hamilton theorem, $q(C) = O$. Equivalently, $C^3 = -3C$. It follows that $C^{n+3} = -3C^{n+1}$ for any integer $n \geq 0$. Then

$$C^9 = -3C^7 = -3(-3C^5) = 9C^5 = 9(-3C^3) = -27C^3 = -27(-3C) = 81C = \begin{pmatrix} 0 & -81 & -81 \\ 81 & 0 & 81 \\ 81 & -81 & 0 \end{pmatrix}.$$

Alternatively, the matrix C^9 can be computed directly by matrix multiplication:

$$C^2 = CC = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix},$$

$$C^4 = C^2C^2 = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix} \begin{pmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 6 & -3 & 3 \\ -3 & 6 & 3 \\ 3 & 3 & 6 \end{pmatrix},$$

$$C^8 = C^4C^4 = \begin{pmatrix} 6 & -3 & 3 \\ -3 & 6 & 3 \\ 3 & 3 & 6 \end{pmatrix} \begin{pmatrix} 6 & -3 & 3 \\ -3 & 6 & 3 \\ 3 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 54 & -27 & 27 \\ -27 & 54 & 27 \\ 27 & 27 & 54 \end{pmatrix},$$

$$C^9 = C^8C = \begin{pmatrix} 54 & -27 & 27 \\ -27 & 54 & 27 \\ 27 & 27 & 54 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -81 & -81 \\ 81 & 0 & 81 \\ 81 & -81 & 0 \end{pmatrix}.$$