Section 9.1 – Basic Notions

The fundamental building blocks of geometry are points, lines, and planes. These terms are not formally defined, but we can describe them intuitively.

Linear Notions

- **Collinear points** are points on the same line.
  - Any two points are collinear but any three points may or may not be.

- Point C is *between* points A and B on line l if C ≠ A, C ≠ B, and C lies on the part of the line connecting A and B.

- A *line segment* is a subset of a line that contains two points of the line and all points between those two points. Notation:

- A *ray* is a subset of a line that contains the endpoint and all points on the line on one side of the point. Notation:

Planar Notions

- Points in the same plane are *coplanar*.

- *Noncoplanar points* cannot be placed in a single plane.

- Lines in the same plane are *coplanar lines*.

- *Skew lines* are lines that do not intersect, and there is no plane that contains them.

- *Intersecting lines* are two coplanar lines with exactly one point in common.

- Two distinct coplanar lines m and n that have no points in common are *parallel lines*. Notation:
Example: How many different lines can be drawn through two points?

Example: Can skew lines be parallel? Why or why not?

Properties of Points, Lines, and Planes

1. There is exactly one line that contains any two distinct points.
2. If two points lie in a plane, then the line containing the points lies in the plane.
3. If two distinct planes intersect, then their intersection is a line.
4. There is exactly one plane that contains any three distinct noncollinear points.
5. A line and a point not on the line determine a plane.
6. Two parallel lines determine a plane.
7. Two intersecting lines determine a plane.

Example: Euclid assumed the first four of the preceding properties as axioms (statements presumed true). Show that properties 5-7 follow logically from the first four properties.
Other Planar Notions

- Two distinct planes either intersect in a line or are parallel.

- A line and a plane can be related in one of three ways:
  - If a line and a plane have no point in common, the line is parallel to the plane.
  - If two points of a line are in the plane, then the entire line is contained in the plane.
  - If a line intersects the plane but is not contained in the plane, it intersects the plane at exactly one point.

- If a line is contained in a plane, it separates the plane into two half-planes. The plane is the union of three disjoint sets: the two half-planes, and the line that separates them.

Angles & Angle Measurement

- When two rays share an endpoint, an angle is formed. The rays of an angle are the sides of the angle, and the common endpoint is the vertex of the angle. Notation:

  - Adjacent angles share a common vertex and a common side and do not have overlapping interiors.

- An angle is measured according to the amount of “opening” between its sides. The degree is commonly used to measure angles. A complete rotation about a point has a measure of 360 degrees, written $360^\circ$. One degree is $\frac{1}{360}$ of a complete rotation. Use a protractor to measure angles.

  - A degree is subdivided into 60 equal parts, minutes, and each minute is divided into 60 equal parts, seconds.

Example: Convert $8.35^\circ$ to degrees, minutes, and seconds.

Example: Convert $42^\circ18'45''$ to decimal degrees.
Types of Angles

- An acute angle has measure less than 90°.
- A right angle has measure 90°.
- An obtuse angle has measure between 90° and 180°.
- A straight angle has measure 180°.

Perpendicular Lines

- When two lines intersect to form right angles, the lines are perpendicular. Two intersecting segments and/or rays are perpendicular if they lie on perpendicular lines. Notation:

- A line perpendicular to a plane is a line that is perpendicular to every line in the plane through its intersection with the plane.

Example: Is it possible for a line intersecting a plane to be perpendicular to two distinct lines in a plane going through its point of intersection with the plane, and yet not be perpendicular to the plane?

Example: Can a line be perpendicular to infinitely many lines?

Theorem 9-1

A line perpendicular to two distinct lines in the plane through its intersection with the plane is perpendicular to the plane.
Section 9.2 – Polygons

Draw a path on a piece of paper without lifting the pencil and without retracing any part of the path except single points. The drawing is called a curve.

- A *simple curve* is one that does not intersect itself, except the starting and stopping points may be the same.

- A *closed curve* is drawn by starting and stopping at the same point.

- A *convex curve* is a simple, closed curve in which a segment connecting any two points in the interior of the curve is wholly contained in the interior.

- A *concave curve* is simple, closed, and not convex.

- A *polygon* is a simple, closed curve made by joining line segments. Notation:

- A polygon and its interior make up a **polygonal region**.

*Example:* Classify each of the following curves.

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<thead>
<tr>
<th>Number of Sides</th>
<th>Name</th>
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<tbody>
<tr>
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</tbody>
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More About Polygons

- Any two sides of a polygon having a common vertex determine an \textit{interior angle} or \textit{angle of the polygon}.
- A side of the polygon and the extension of a contiguous side of the polygon determine an \textit{exterior angle} of a convex polygon.
- A \textit{diagonal} is a line segment connecting nonconsecutive vertices of a polygon.

Congruent Segments and Angles

- \textit{Congruent segments} have the same length.
- \textit{Congruent angles} have the same measure.
- Notation:

Regular Polygons

A \textit{regular polygon} is one in which all the interior angles are congruent and all the sides are congruent. A regular polygon is both equiangular and equilateral.

Triangles Classifications

- A \textit{right triangle} contains one right angle.
- An \textit{acute triangle} has all acute angles.
- An \textit{obtuse triangle} contains one obtuse angle.
- A \textit{scalene triangle} has no congruent sides.
- An \textit{isosceles triangle} has at least two congruent sides.
- An \textit{equilateral triangle} has three congruent sides.
Quadrilateral Classifications

- A trapezoid is a quadrilateral with at least one pair of parallel sides. (Some elementary texts define a trapezoid as a quadrilateral with exactly one pair of parallel sides.)

- An isosceles trapezoid is a trapezoid with exactly one pair of congruent sides. (Equivalently, an isosceles trapezoid is a trapezoid with two congruent base angles.)

- A kite is a quadrilateral with two adjacent sides congruent and the other two sides also congruent.

- A parallelogram is a quadrilateral in which each pair of opposite sides is parallel.

- A rectangle is a parallelogram with a right angle. (Equivalently, a rectangle is a quadrilateral with four right angles.)

- A rhombus is a parallelogram with two adjacent sides congruent. (Equivalently, a rhombus is a quadrilateral with all sides congruent.)

- A square is a rectangle with two adjacent sides congruent. (Equivalently, a square is a quadrilateral with four right angles and four congruent sides.)
Example: Determine whether each of the following statements is true or false.

- An equilateral triangle is isosceles.

- A square is a regular quadrilateral.

- If one angle of a rhombus is a right angle, then all the angles of the rhombus are right angles.

- A rectangle is a kite.

- A square is a rhombus with a right angle.

- All the angles of a rectangle are right angles.

- A rectangle is an isosceles trapezoid.

- If a kite has a right angle, then it must be a square.
Section 9.3 – More About Angles

- *Vertical angles* are pairs of angles opposite each other at the intersection of two lines.
- *Supplementary angles* are two angles, the sum of whose measures is 180°. Each angle is a supplement of the other.
- *Complementary angles* are two angles, the sum of whose measures is 90°. Each angle is a complement of the other.
- A line that intersects a pair of lines in a plane is a *transversal* of those lines.
  - Vertical Angles
  - Interior Angles
  - Exterior Angles
  - Alternate Interior Angles
  - Alternate Exterior Angles
  - Corresponding Angles

**Theorem 9-2** Vertical angles are congruent.

*Example:* Prove Theorem 9-2.
**Theorem 9-3**
If any two distinct coplanar lines are cut by a transversal, then a pair of corresponding angles, alternate interior angles, or alternate exterior angles are congruent if and only if the lines are parallel.

**The Sum of the Measures of the Interior Angles of a Convex Polygon with \( n \) Sides**

**Theorem 9-4**
The sum of the measures of the interior angles of a triangle is 180°.

*Example:* Prove Theorem 9-4.

**Theorem 9-5**
The sum of the measures of the exterior angles (one at each vertex) of a convex polygon is 360°.

*Example:* Show that Theorem 9-5 holds true for a pentagon.
Theorem 9-6

a. The sum of the measures of the interior angles of any convex polygon with \( n \) sides is 
\[ 180n - 360, \text{ or } (n - 2)180^\circ. \]

b. The degree measure of a single interior angle of a regular \( n \)-gon is 
\[ \frac{180n - 360}{n}, \text{ or } \frac{(n - 2)180}{n}. \]

Example: Prove Theorem 9-6a.

Example: What is the measure of an interior angle of a regular
a. quadrilateral?

b. hexagon?

c. nonagon?

Example: The ratio of the interior angles of a pentagon is 1 : 2 : 2 : 3 : 4. What is the measure of the largest angle?
Section 9.4 – Geometry in Three Dimensions

Simple Closed Surfaces

- A **simple closed surface** has exactly one interior, no holes, and is hollow.
- A **solid** is a simple closed surface with all interior points.
- A **sphere** is the set of all points at a given distance from a given point, the **center**.
- A **polyhedron** is a simple closed surface made up of polygonal regions, or **faces**.
  - The vertices of the polygonal regions are the **vertices** of the polyhedron, and the sides of each polygonal region are the **edges** of the polyhedron.
- A **prism** is a polyhedron in which two congruent faces lie in parallel planes and the other faces are bounded by parallelograms.
  - The parallel faces of a prism are the **bases** of the prism. A prism is usually named after its bases. The faces other than the bases are the **lateral faces** of a prism.
  - A **right prism** is one in which the lateral faces are all bounded by rectangles.
  - An **oblique prism** is one in which some of the lateral faces are not bounded by rectangles.
- A **pyramid** is a polyhedron determined by a polygon and a point not in the plane of the polygon.
  - The polygonal region is the **base** of the pyramid, and the point is the **apex**. The faces other than the base are **lateral faces**.
  - A **right pyramid** is one in which all the lateral faces are congruent isosceles triangles.

*Example:* Draw a right pentagonal prism.

*Example:* Draw a pentagonal pyramid.
Regular Polyhedra

- A polyhedron is a *convex* polyhedron if and only if the segment connecting any two points in the interior of the polyhedron is itself in the interior.
- A *regular polyhedron* is a convex polyhedron a) whose faces are congruent regular polygonal regions, and b) the number of edges that meet at each vertex is the same for all the vertices of the polyhedron.

*Example:* How many regular polyhedra are there?
Cylinders and Cones

- A **cylinder** is the surface formed by moving a segment (keeping it parallel to the original segment) to form a simple closed non-polygonal curve at its ends, along with the simple closed curves, and their interiors.
  - The simple closed curves traced by the endpoints of the segment, along with their interiors, are the **bases** of the cylinder; the remaining points constitute the **lateral surface** of the cylinder.
  - A **circular cylinder** is one whose base is a circular region.
  - A **right cylinder** is one in which the line segment forming the cylinder is perpendicular to a base.
  - An **oblique cylinder** is any cylinder that is not a right cylinder.

- A **cone** is the union of line segments connecting a point to each point of a simple closed curve, not coplanar with the point, the simple closed curve, and the interior of the curve.
  - The point is the **vertex** of the cone; the points not in the base constitute the **lateral surface** of the cone. A line segment from the vertex perpendicular to the plane of the base is the **altitude** of the cone.
  - A **right circular cone** is one whose altitude intersects the circular base at the center of the circle.
  - An **oblique circular cone** is a circular cone that is not a right circular cone.
Section 9.5 – Networks

Königsberg Bridge Problem

Is it possible to walk across all the bridges so that each bridge is crossed exactly once on the same walk?

The problem can be made simpler by representing the problem in a network. Let the bridges represent vertices and the paths between bridges represent arcs, sometimes also called edges.

Using the network, the problem can be restated:
Is there a path through the network beginning at some vertex and ending at the same or another vertex such that each arc is traversed exactly once?
Properties of Networks
- A network that has a path such that each arc is passed through exactly once is traversable or an Euler path.
- A network that is traversable in such a way that the starting point and the ending point are the same is an Euler circuit.
- If the number of arcs meeting at a vertex is even, it is known as an even vertex.
- If the number of arcs meeting at a vertex is odd, it is known as an odd vertex.

Determining if a network is traversable
1. If a network has all even vertices, it is traversable. Any vertex can be the starting point, and the same vertex must be the stopping point. This type of network is known as an Euler circuit.
2. If a network has exactly two odd vertices, it is traversable. One odd vertex must be the starting point, and the other odd vertex must be the stopping point.
3. If a network has exactly one or more than two odd vertices, it is not traversable.

Example: Does the Königsberg bridge problem represent a traversable network? Why or why not?

Example: Which of the following networks are traversable? If a network is traversable, find a path for the network and determine if it is an Euler circuit.
Example: The following network is not an Euler circuit.

a. Add exactly two arcs to the network so that the resulting network will be an Euler circuit.

b. Add exactly one arc so that the resulting network will be an Euler circuit.