5. Exact Equations (section 2.6)

1. Consider a first order differential equation

$$P(x,y) + Q(x,y)y' = 0 \quad (\text{or} \quad P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = 0) \tag{1}$$

with P and Q having continuous partial derivatives in the region R of $\mathbb{R}^2$.

DEFINITION 1. *The equation* (1) *is called* exact *on the region R if there exists a differentiable function function Φ on the region $\mathbb{R}$ such that*

$$\begin{cases} \Phi_x(x,y) = P(x,y) \\ \Phi_y(x,y) = Q(x,y) \end{cases} \tag{2}$$

for every $(x,y) \in R$.

REMARK 2. *Note that exact equations are very special: system* (2) *is the system of two equations for one unknown function Φ and it does not have a solution for an arbitrary choice of the pair of functions (P,Q). In order that the system* (2) *will have a solution the pair (P,Q) must satisfy a certain compatibility condition (see equation* (3) *below)*

2. Assume that Φ satisfies the system (2). Then $y(x)$ is a solution of the equation (1) if and only if

$$\frac{d}{dx}\Phi\big(x, y(x)\big) =$$

In other words, an exact equation (1) represents the **exact differential** of the function $\Phi(x,y)$., i.e $d\Phi = 0$ (this is the origin of the word exact).

Therefore,

$$\Phi\big(x, y(x)\big) \equiv C$$

for some constant C.

CONCLUSION *Any integral curve $y = y(x)$ of an exact equation* (1) *lies on a level curve of the function $\Phi(x,y)$ and the general solution of equation* (1) *is given in an implicite form by*

$$\boxed{\Phi(x,y) = C}$$

3. *EFFECTIVE TEST for Exactness:* If ODE (1) is exact on the region R, then

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}} \quad \text{on } R \tag{3}$$

Proof.

□

PROPOSITION 3. *If the region R is simply connected (i.e. without holes, for example $R = \mathbb{R}^2$) then ODE (1) is exact on R if and only if the relation (3) holds on R.*

4. *Determine whether the following ODE are exact:*

   (a) $3x^2 - 2xy + 2 + (6y^2 - x^2 + 3)y' = 0$

   (b) $(3x^2y + 2xy + y^3)\mathrm{d}x + (x^2 + y^2)\mathrm{d}y = 0$

REMARK 4. If you are asked to verify exactness you are not allowed to multiply (or divide) you equation by a nonconstant function: such operation will destroy the exactness.

5. *Relation to the topic of "Conservative Vector Fields" from Calculus-3 (Stewart, Section 14.3)* Consider the vector field $\vec{F} = \langle P, Q\rangle$ in the region R . The system (2) is equivalent to the equation

$$\operatorname{grad}\Phi = \vec{F}. \tag{4}$$

In other words, vector fields $\vec{F}$ is conservative, the function Φ is a potential of $\vec{F}$ and (3) is the test to check whether $\vec{F}$ is conservative in a simply connected region R.

CONCLUSION: *A general solution to an exact differential equation can be found by the method used in Calculus 3 to find a potential function for a conservative vector field in a plane region.*

6. Solve $3x^2 - 2xy + 2 + (6y^2 - x^2 + 3)y' = 0$.