Solution of item 5 of section 3 of lecture notes regarding the method of integrating factor

5. Consider

$$y' - 3xy = -xe^{x^2}. \tag{1}$$

1. Find the general solution.

   Solution

   In our case $p(x) = -3x$ and $g(x) = -xe^{x^2}$. Therefore, an integrating factor satisfies

   $$\mu' = -3x\mu.$$

   So, we can take an integrating factor as

   $$\mu(x) = e^{\int(-3x)\,dx} = e^{-\frac{3x^2}{2}}.$$

   Multiply both sides of (1) by this $\mu(x)$. Then

   $$\left(e^{-\frac{3x^2}{2}}y\right)' = -xe^{x^2}e^{-\frac{3x^2}{2}} = -xe^{-\frac{x^2}{2}}.$$

   Integrate both parts with respect to x:

   $$e^{-\frac{3x^2}{2}}y = \int -xe^{-\frac{x^2}{2}}\,dx = (*)$$

   Use u-substitution: $u = -\dfrac{x^2}{2}$. Then $du = -xdx$ and

   $(*) = \int e^u\,du = e^u + C = e^{-\frac{x^2}{2}} + C.$

   Therefore,

   $$e^{-\frac{3x^2}{2}}y = e^{-\frac{x^2}{2}} + C$$

   Finally, multiplying both part by $e^{\frac{3x^2}{2}}$ we get

   $$y(x) = e^{\frac{3x^2}{2}}(e^{-\frac{x^2}{2}} + C) = \boxed{e^{x^2} + Ce^{\frac{3x^2}{2}}} \tag{2}$$

2. Find the solution satisfying the initial condition $y(0) = y_0$.

   emphSolution

   Plug $t = 0$ and $y = y_0$ into general solution (2) to determine C:

   $$y_0 = 1 + C \Rightarrow C = y_0 - 1$$

   So,

   $$\boxed{y(x) = e^{x^2} + (y_0 - 1)e^{\frac{3x^2}{2}}} \tag{3}$$

3. How do the solutions behave as x becomes large (i.e $x \to +\infty$)? Does that behavior depend on the choice of the initial value y_0?

   emphSolution

   The term $e^{\frac{3x^2}{2}}$ dominates the term $e^{\frac{x^2}{2}}$ as x approach infinity so everything will depend on the sign of the coefficient $y_0 - 1$ near the dominating term:

   - If $y_0 > 1$, then $y(x) \to +\infty$ as $x \to +\infty$;
   - If $y_0 < 1$, then $y(x) \to -\infty$ as $x \to +\infty$;
   - The transition from one behavior to another one occurs as $y_0 = 1$. In this case $y(x) = e^{x^2} \to +\infty$ as $x \to +\infty$.

   So, the behaviour depends on the initial conditions.