8. Autonomous and non-autonomous first order differential equations and systems , vector fields and direction fields (sections 1.1, 7.1)

1. Consider the first order ODE

$$\frac{dy}{dt} = F(t, y). \tag{1}$$

We work in much more general setting than in section 1.1: $y(t)$ is in $\mathbb{R}^n$, $y(t) = (y_1(t), \dots, y_n(t))$, F is a vector function, consisting of n functions of $n+1$ variables $(t, y_1, \dots, y_n)$,

$$F(t, y) = \big(F_1(t, y_1, \dots, y_n), \dots, F_n(y_1, \dots, y_n)\big),$$

so that (1) is a system of n differential equations of first order with unknown function $y_1(t), \dots y_n(t)$:

DEFINITION 1. *The equation/system* (1) *is called* autonomous *if the right-hand side of it is independent of* t*, i.e. is of the form* $F(y)$*,*

$$\frac{dy}{dt} = F(y). \tag{2}$$

and non-autonomous otherwise.

REMARK 2. *Autonomous equations(systems) are special first order equations (systems). However, any non-autonomous system on* n *unknown functions* $(y_1, \dots, y_n)$ *of* t *can be seen as an autonomous system in* $n+1$*-unknown functions as follows:*

2. **Vector fields and autonomous first order systems:**

   (a) A *vector field* F on $\mathbb{R}^n$: at each point y of $\mathbb{R}^n$ a vector $F(y)$ starting at this point y is given.

   (b) An *integral curve* $y(t)$ *of a vector field* F is a curve such that the velocity $y'(t)$ to it at every its point $y(t)$ (or, equivalently, at every time momemt t) coincides with the vector fields F at this point, i.e. with the vector $F(y(t))$.

   In other words,

$$y'(t) = F(y(t))$$

   i.e. $y(t)$ is an integral curve of the field F if and only of it is a solution of the autonomous equation (2)

3. **Directional fields and (non-autonomous) first order systems:**

   (a) A *direction field on* $\mathbb{R}^{n+1}$ with coordinates (t, y), where $y \in R^n$: at each point of $\mathbb{R}^{n+1}$ a straight line passing through this point is given such that a vector generated this line has non-zero t-component.

   Hence this generating vectors can be given in the form $(1, F(t, y))$.

   (b) An *integral curve of a direction field* is a curve in $\mathbb{R}^{n+1}$ of the form $(t, y(t))$ such that the velocity to this curve at every its point $\big(t, y(t)\big)$ is in the direction of the straight line of the direction field given at this point.

   So, if the lines of the direction field are generated by vectors of the form $(1, F(t, y))$, then the curve $(t, y(t))$ is an integral curve of this direction field if and only if

$$y'(t) = F(t, y(t)),$$

   i.e. $y(t)$ is a solution of the system (1).

   The field of directions generated by vectors of the form $(1, F(t, y))$ is called the *direction field of the first order equation/system* (1).

4. Case $n = 1$, i.e. of the first order equation

   (a) in this case $y = y(t)$ is a solution of the ODE $y' = F(t, y)$ if and only if at each point $(t, y(t))$ of this curve the tangent line to the graph of $y(t)$ has slope $F(t, y)$.

   So, the direction field of the first order ODE (1) is the direction field such that the line at the point (t, y) has a slope $F(t, y)$.

   (b) To sketch direction field use the following steps:

   - Choose a rectangular grid of points in the ty-plane.
   - Calculate the slopes of tangent lines to the integral curves at the gridpoints.
   - Draw a short line segment of the tangent lines through the gridpoints.

   Note: More gridpoints $\Longrightarrow$ better description of integral curves (general shape of solution).

   (c) *FALLING HAILSTONE* Consider hailstone with mass $m = 0.03kg$ and drag coefficient $\gamma = 0.006kg/s$

   i. Write down the ODE describing the motion of the hailstone.

   ii. Sketch the direction field

   iii. We can use Matlab to get directional field:

```
[t,v]=meshgrid(0:0.4:8, 30:1.2:60);
S=9.8-0.2*v;
quiver(t,v,ones(size(S)),S), axis tight
```