Page 1) Extra class of 11/30/2017 MATH 308

1) More examples on Laplace transform of step functions

Example 1 Find the Laplace transform of the function

$$f(t) = \begin{cases} t^2-1, & 0\le t<2 \\ -t^2+2, & 2\le t<4 \\ -t^2+3t, & 4\le t \end{cases}$$

Solution

Step 1 Represent f using step functions

$$f(t) = (t^2-1) + (-t^2+2-(t^2-1))\,u_2(t) + ((-t^2+3t)-(-t^2+2))\,u_4(t) = (t^2-1) + (-2t^2+3)\,u_2(t) + (3t-2)\,u_4(t)$$

Step 2 Find the Laplace transform of each term

1) $\mathcal{L}\{t^2-1\} = \dfrac{2}{s^3} - \dfrac{1}{s}$

2) $\mathcal{L}\{(-2t^2+3)\,u_2(t)\} = e^{-2s}\,\mathcal{L}\{\underbrace{-2(t+2)^2+3}_{-2(t^2+4t+4)+3=}\} =$

$= e^{-2s}\, \mathcal{L}\{-2t^2-8t-5\} = -e^{-2s}\left(\frac{4}{s^3}+\frac{8}{s^2}+\frac{5}{s}\right)$

3) $\mathcal{L}\{(3t-2)\, u_4(t)\} = e^{-4s}\, \mathcal{L}\{\underbrace{3(t+4)-2}_{3t+10}\} =$

$= e^{-4s}\left(\frac{3}{s^2}+\frac{10}{s}\right)$

Combining all together we get

$\frac{2}{s^3}-\frac{1}{s}-e^{-2s}\left(\frac{4}{s^3}+\frac{8}{s^2}+\frac{5}{s}\right)+e^{-4s}\left(\frac{3}{s^2}+\frac{10}{s}\right)$

<u>Example 2</u> Find the inverse Laplace transform of the function $F(s) = \dfrac{e^{-\pi s}(s^2+5s+4)}{(s-1)(4s^2+8s+13)}$

<u>Solution</u> 1) First find the inverse Laplace of

$$\frac{s^2+5s+4}{(s-1)(4s^2+8s+13)}$$

$4s^2+8s+13 = 4\left(s^2+2s+\frac{13}{4}\right)$ completing squares

$= 4\left(\underbrace{s^2+2s+1}_{(s+1)^2}+\frac{9}{4}\right) = 4\left((s+1)^2+\left(\frac{3}{2}\right)^2\right)$ $\alpha=-1$, $\beta=\frac{3}{2}$

So

$$\frac{s^2+5s+4}{(s-1)(4s^2+8s+13)} = \frac{s^2+5s+4}{4(s-1)\left((s+1)^2+\left(\frac{3}{2}\right)^2\right)}$$

Find the partial fraction decomposition of

$$\frac{s^2+5s+4}{(s-1)\left((s+1)^2+\left(\frac{3}{2}\right)^2\right)} = \frac{A}{s-1} + \frac{B(s+1)+C\cdot\frac{3}{2}}{(s+1)^2+\left(\frac{3}{2}\right)^2}$$

Common denominator:

$$s^2+5s+4 = A\left((s+1)^2+\left(\frac{3}{2}\right)^2\right) + \left(B(s+1)+C\cdot\frac{3}{2}\right)(s-1)$$

If $s=1$. $\underbrace{1+5+4}_{10} = A\left(\underbrace{2^2+\left(\frac{3}{2}\right)^2}_{4+\frac{9}{4}}\right) = A\cdot\frac{25}{4} \Rightarrow$

$$A = \frac{40}{25} = \frac{8}{5}$$

If $s=-1$. $\underbrace{1-5+4}_{0} = A\cdot\frac{9}{4} + C\cdot\frac{3}{2}(-2) \Rightarrow$

$$\frac{\cancel{8}^{2}}{5}\cdot\frac{9}{\cancel{4}} - 3C = 0 \Rightarrow C = \frac{6}{5}$$

Coefficient of s^2: $1 = A+B \Rightarrow B = 1-\frac{8}{5} = 1-\frac{8}{5} = -\frac{3}{5}$

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So $$\frac{s^2+5s+4}{(s-1)(4s^2+8s+13)} = \frac{1}{4}\left(\frac{8}{5}\frac{1}{s-1} - \frac{3}{5}\frac{s+1}{(s+1)^2+\left(\frac{3}{2}\right)^2} + \frac{6}{5}\frac{\frac{3}{2}}{(s+1)^2+\left(\frac{3}{2}\right)^2}\right) = \frac{2}{5}\frac{1}{s-1} - \frac{3}{20}\frac{s+1}{(s+1)^2+\left(\frac{3}{2}\right)^2} + \frac{3}{10}\frac{\frac{3}{2}}{(s+1)^2+\left(\frac{3}{2}\right)^2} \Rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{s^2+5s+4}{(s-1)(4s^2+8s+13)}\right\} = \frac{2}{5}e^{t} - \frac{3}{20}e^{-t}\cos\left(\frac{3}{2}t\right) + \frac{3}{10}e^{-t}\sin\left(\frac{3}{2}t\right) \quad (*)$$

2) Then $$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}(s^2+5s+4)}{(s-1)(4s^2+8s+13)}\right\} = u_\pi(t)\left(\frac{2}{5}e^{t-\pi} - \frac{3}{20}e^{-(t-\pi)}\cos\left(\frac{3}{2}(t-\pi)\right) + \frac{3}{10}e^{-(t-\pi)}\sin\left(\frac{3}{2}(t-\pi)\right)\right) =$$

multiply (*) by $u_\pi(t)$ and replace t by $t-\pi$ everywhere in (*)

$$= u_\pi(t)\left(\frac{2}{5}e^{-\pi}e^{t} - \frac{3}{20}e^{\pi}e^{-t}\underbrace{\cos\left(\frac{3}{2}t-\frac{3}{2}\pi\right)}_{-\sin\frac{3}{2}t} + \frac{3}{10}e^{\pi}e^{-t}\underbrace{\sin\left(\frac{3}{2}t-\frac{3\pi}{2}\right)}_{\cos\frac{3}{2}t}\right)$$

$$\boxed{= u_\pi(t)\left(\frac{2}{5}e^{-\pi}e^{t} + \frac{3}{20}e^{\pi}e^{-t}\sin\frac{3}{2}t + \frac{3}{10}e^{\pi}e^{-t}\cos\frac{3}{2}t\right)}$$

Rem General rule how to transform $\cos/\sin\left(\frac{\pi n}{2} \pm \theta\right)$ to $\pm \frac{\cos}{\sin}(\theta)$.

1) If n is odd then cos is switched to sin and sin is switched to cos, while if n is even the function does not change.

2) To determine the sign (+ or −) put $\frac{\pi n}{2}$ on the unit circle and assume that θ is small and positive $(0<\theta<\frac{\pi}{2})$. Then if for example you look for $\cos\left(\frac{\pi n}{2} \pm \theta\right)$ check what is the sign of $\cos\left(\frac{\pi n}{2} \pm \theta\right)$ for such θ. Then it will give the right sign in the formula.

2) Convolution integral

Example: Find the inverse Laplace transform of $\frac{s}{(s^2+49)(s^2-6s+34)}$

Take $F = \frac{s}{s^2+49}$, $G = \frac{1}{s^2-6s+10} = \frac{1}{(s-3)^2+\underbrace{25}_{5^2}}$

$\underbrace{49}_{7^2}$

1) $f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+7^2}\right\} \overset{\text{table}}{=} \cos(7t)$

2) $g(t) = \mathcal{L}^{-1}\left(\frac{1}{(s-3)^2+5^2}\right) = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{5}{(s-3)^2+5^2}\right\} \overset{\text{table}}{=}$

$= \frac{1}{5}e^{3t}\sin 5t$

by the convolution theorem

$\mathcal{L}^{-1}\{F \cdot G\} = f * g(t) = \underbrace{\frac{1}{5}\int_0^t \cos 7(t-\tau)\, e^{3\tau}\sin 5\tau\, d\tau}$

some people stop here in their homework but you need to continue to calculate the integral

$= \frac{1}{5}\int_0^t e^{3\tau}\cos\underbrace{7(t-\tau)}_{\beta}\sin\underbrace{5\tau}_{\alpha}\, d\tau$

$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right) \Rightarrow$

$\sin\underbrace{5\tau}_{\alpha}\cos\underbrace{7(t-\tau)}_{\beta} = \frac{1}{2}\left(\sin(5\tau+7t-7\tau)+\right.$

$\left.+\sin(5\tau-7t+7\tau)\right) = \frac{1}{2}\left(\sin(7t-2\tau)+\right.$

$\left.+\frac{1}{2}\sin(12\tau-7t)\right)$

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$$\frac{1}{5}\int_0^t e^{3\tau}\cos 7(t-\tau)\sin 5\tau \, d\tau =$$

$$= \frac{1}{10}\int_0^t e^{3\tau}\sin(7t-2\tau) + \frac{1}{10}\int_0^t e^{3t}\sin(12\tau-7t))$$

$$1)\ \int_0^t e^{3\tau}\sin(7t-2\tau)\,d\tau = \frac{1}{3}e^{3t}\sin(7t-2\tau) +$$

(integration by parts ~~using logs~~)

$$+\frac{2}{3}\int_0^t e^{3t}\cos(7t-2\tau)\,d\tau = \frac{1}{3}e^{3\tau}\sin(7t-2\tau)\Big|_0^t +$$

$$+\frac{2}{9}e^{3\tau}\cos(7t-2\tau)\Big|_0^t - \frac{4}{9}\int_0^t e^{3t}\sin(7t-2\tau)\,d\tau \Rightarrow$$

$$\left(1+\frac{4}{9}\right)\int_0^t e^{3\tau}\sin(7t-2\tau)\,dt = \frac{1}{3}e^{3\tau}\sin(7t-2\tau)\Big|_0^t +$$

$$+\frac{2}{9}e^{3\tau}\cos(7t-2\tau)\Big|_0^t \Rightarrow$$

$$\int_0^t e^{3\tau}\sin(7t-2\tau)\,d\tau = \frac{9}{13}\left(\frac{1}{3}\left(e^{3t}\sin 5t - \sin 7t\right)\right.$$

$$\left.+\frac{2}{9}\left(e^{3t}\cos 5t - \cos 7t\right)\right) = \frac{3}{13}e^{3t}\sin 5t - \frac{3}{13}\sin 7t +$$

$$+\frac{2}{13}e^{3t}\cos 5t - \frac{2}{13}\cos 7t$$

$$\int_0^t e^{3\tau}\sin(12\tau-7t)\,d\tau = \frac{1}{3}e^{3t}\sin(12\tau-7t)\Big|_0^t -$$

$$-\underbrace{\frac{12}{3}}_{4}\int_0^t e^{3\tau}\cos(12\tau-7t)\,d\tau = \frac{1}{3}e^{3\tau}\sin(12\tau-7t)\Big|_0^t$$

$$-4\left(\frac{1}{3}e^{3\tau}\cos(12\tau-7t)\Big|_0^t + 4\int_0^t e^{3\tau}\sin(12\tau-7t)\,d\tau\right.$$

$$= \frac{1}{3}e^{3\tau}\sin(12\tau-7t)\Big|_0^t - \frac{4}{3}e^{3\tau}\cos(12\tau-7t)\Big|_0^t -$$

$$-16\int_0^t e^{3\tau}\sin(12\tau-7t)\,d\tau \Rightarrow$$

$$17\int_0^t e^{3\tau}\sin(12\tau-7t)\,d\tau = \frac{1}{3}e^{3t}\sin 5t - \frac{1}{3}\underbrace{\sin(-7t)}_{-\sin 7t}$$

$$-\frac{4}{3}e^{3t}\cos 5t + \frac{4}{3}\underbrace{\cos(-7t)}_{\cos 7t} \Rightarrow$$

$$\int_0^t e^{3\tau}\sin(12\tau-7t)\,d\tau = \frac{1}{51}e^{3t}\sin 5t + \frac{1}{51}\sin 7t -$$

$$-\frac{4}{51}e^{3t}\cos 5t + \frac{4}{51}\cos 7t.$$

Combining all together we get

$$\mathcal{L}^{-1}\{F\cdot G\} = \frac{1}{10}\left(\frac{3}{13}e^{3t}\sin 5t - \frac{3}{13}\sin 7t + \right.$$

$$+\frac{2}{13}e^{3t}\cos 5t - \frac{2}{13}\cos 7t + \frac{1}{51}e^{3t}\sin 5t + \frac{1}{51}\sin 7t -$$

$$\left. -\frac{4}{51}e^{3t}\cos 5t + \frac{4}{51}\cos 7t\right) =$$

$$= \frac{1}{10}\left(\underbrace{\left(\frac{3}{13}+\frac{1}{51}\right)}_{\frac{166}{663}}e^{3t}\sin 5t + \underbrace{\left(\frac{2}{13}-\frac{4}{51}\right)}_{\frac{102-52}{663}=\frac{50}{663}}e^{3t}\cos 5t\right.$$

$$\left. + \underbrace{\left(-\frac{2}{13}+\frac{4}{51}\right)}_{-\frac{50}{663}}\cos 7t + \underbrace{\left(-\frac{3}{13}+\frac{1}{51}\right)}_{\frac{13-153}{663}=-\frac{140}{663}}\sin 7t\right) =$$

$$= \boxed{\frac{166}{3315}e^{3t}\sin 5t + \frac{5}{663}e^{3t}\cos 5t - \frac{5}{663}\cos 7t - \frac{14}{663}\sin 7t}$$

Example 4

(a) Express the solution of the given initial value problem in terms of a convolution integral

$$y'' + 4y' + 3y = g(t), \quad y(0) = 1, \quad y'(0) = -2$$

Solution $3\times \quad \mathcal{L}\{y\} = Y$

$+ \quad 4\times \quad \mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 1$

$+ \quad 1\times \quad \mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) =$

$= s^2Y(s) - s + 2$

$$\mathcal{L}\{y'' + 4y' + 3y\} = (s^2+4s+3)Y - s + 2 - 4 =$$

$$= (s^2+4s+3)Y - s - 2 = G(s) \Rightarrow$$

$$Y(s) = \frac{s+2}{s^2+4s+3} + \frac{1}{s^2+4s+3} \cdot G(s)$$

1) $\dfrac{s+2}{(s^2+4s+3)} = \dfrac{s+2}{(s+1)(s+3)} = \dfrac{A}{s+1} + \dfrac{B}{s+3} \Rightarrow$

$$(s+2) = A(s+3) + B(s+1)$$

$s=-1: \quad 1 = 2A \Rightarrow A = \frac{1}{2}$

$s=-3: \quad -1 = -2B \Rightarrow B = \frac{1}{2} \Rightarrow$

$$\frac{s+2}{(s+1)(s+3)} = \frac{1}{2}\,\frac{1}{s+1} + \frac{1}{2}\,\frac{1}{s+3} \Rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)(s+3)}\right\} = \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}$$

2) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+3}\,G(s)\right\} = \mathcal{L}^{-1}\left(\frac{1}{s^2+4s+3}\right) * g$

$$\frac{1}{s^2+4s+3} = \frac{1}{(s+1)(s+3)} = \frac{C}{(s+1)} + \frac{D}{s+3} \Rightarrow$$

$$1 = C(s+3) + D(s+1)$$

$s=-1: \quad 1 = 2C \Rightarrow C = \frac{1}{2}$

$s=-3: \quad 1 = -2D \Rightarrow D = -\frac{1}{2}$

$$\frac{1}{s^2+4s+3} = \frac{1}{2}\left(\frac{1}{s+1} - \frac{1}{s+3}\right) \Rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+3}\right\} = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \Rightarrow$$

$\boxed{\text{Page 12}}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+3}\,G(s)\right\} = \left(\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}\right) * g(t) =$$

$$= \frac{1}{2}\int_0^t \left(e^{-t+\tau} - e^{-3t+3\tau}\right) g(\tau)\,d\tau \Rightarrow$$

$$\boxed{y(t) = \frac{1}{2}\int_0^t \left(e^{\tau-t} - e^{3\tau-3t}\right) g(\tau)\,d\tau + \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}}$$

(b) Solve the same problem using the method of variation of parameters and compare the answer with item a i.

$$y'' + 4y' + 3y = g(t)$$

Char. equation is $\lambda^2 + 4\lambda + 3 = 0 \Leftrightarrow \lambda_1 = -1,\ \lambda_2 = -3$

Look for a solution in the form

$y(t) = u_1(t)e^{-t} + u_2(t)e^{-3t}$ so that

$$\begin{pmatrix} e^{-t} & e^{-3t} \\ -e^{-t} & -3e^{-3t} \end{pmatrix}\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \Rightarrow$$

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$$u_1'(t) = \frac{\begin{vmatrix} 0 & e^{-3t} \\ g(t) & -3e^{-3t} \end{vmatrix}}{\begin{vmatrix} e^{-t} & e^{-3t} \\ -e^{-t} & -3e^{-3t} \end{vmatrix}} = \frac{-g\,e^{-3t}}{\underbrace{-3e^{-4t} + e^{-4t}}_{-2e^{-4t}}} =$$

$$= \frac{1}{2} g(t)\, e^{t} \Rightarrow$$

$$u_1(t) = \frac{1}{2}\int_0^t g(\tau) e^{\tau} d\tau + C_1$$

$$u_2'(t) = \frac{\begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & g(t) \end{vmatrix}}{-2e^{-4t}} = \frac{g(t)e^{-t}}{-2e^{-4t}} = -\frac{1}{2} e^{3t} g(t)$$

$$\Rightarrow u_2(t) = -\frac{1}{2}\int_0^t g(\tau) e^{3\tau} d\tau \Rightarrow$$

$$y(t) = u_1(t)\, e^{-t} + u_2(t) e^{-3t} = \frac{1}{2}\left(\int_0^t g(\tau) e^{\tau} d\tau\right) e^{-t} -$$

$$-\frac{1}{2}\left(\int_0^t g(\tau) e^{3\tau} d\tau\right) e^{-3t} + C_1 e^{-t} + C_2 e^{-3t} =$$

$$= \frac{1}{2}\int_0^t \left(e^{\tau-t} - e^{3\tau-3t}\right) dt + C_1 e^{-t} + C_2 e^{-3t}$$

Use initial conditions $y(0)=1$, $y'(0)=-2$

to determine C_1 & C_2:

$y(0)=1 \Rightarrow \quad C_1 + C_2 = 1$

$$y'(0) = \frac{1}{2}\left(\underset{1}{e^{0-0}} - \underset{1}{e^{30-30}}\right) + \underbrace{\int_0^0 \frac{d}{dt}\left(e^{\tau-t} - e^{3\tau-3t}\right)}_{0}$$

$+ C_1 e^{0} - 3C_2 e^{-3t} = -2 \Rightarrow \quad C_1 + 3C_2 = 2$

$\begin{cases} C_1 + C_2 = 1 \\ C_1 + 3C_2 = 2 \end{cases} \Rightarrow$ Eq2 - Eq1: $2C_2 = 1 \Rightarrow C_2 = \frac{1}{2} \Rightarrow$

$C_1 = 1 - C_2 = 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow$

$$\boxed{y(t) = \frac{1}{2}\int_0^t \left(e^{\tau-t} - e^{3\tau-3t}\right) dt + \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}}$$

(⇒ in ⇒)

3) Example of finding fundamental set of solutions using series solutions

Example 5 Find two linearly independent solutions of the equation

$$L[y] = \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2y = 0$$

Take for simplicity $x = x_0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow 2y(x) = \sum_{n=0}^{\infty} 2a_n x^n$$

$$x\frac{dy}{dx} = x\sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^n$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2y = \sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} - 2n a_n - 2a_n\right) x^n = 0 \Rightarrow a_{n+2} = \frac{2\cancel{(n+1)}}{(n+2)\cancel{(n+1)}} a_n \Rightarrow$$

$$a_{n+2} = \frac{2}{n+2} a_n$$

So if n is even, $n=2k$, $k \geq 0$

$$a_{2k} = \frac{2}{2k} a_{2k-2} = \frac{1}{k} a_{2k-2} = \frac{1}{k}\frac{1}{k-1} a_{2k-4} = \cdots$$

$$= \frac{1}{k(k-1)\cdots 2\cdot 1} a_0 = \frac{1}{k!} a_0$$

So $a_{2k} = \frac{1}{k!} a_0$

If n is odd, $n = 2k+1$, $k \geq 0$

$$a_{2k+1} = \frac{2}{2k+1} a_{2k-1} = \frac{2}{2k+1}\cdot\frac{2}{2k-1} a_{2k-3} = \cdots$$

$$= \frac{2^k}{\underbrace{(2k+1)(2k-1)\cdots 3}_{(2k+1)!!}} a_1 \Rightarrow a_{2k+1} = \frac{1}{(2k+1)!!} a_1$$

$(2k+1)!! \rightarrow$ product of all odd numbers up to $2k+1$

So if $a_0 = 1$ and $a_1 = 0$ $(\Leftrightarrow y(0) = 1, y'(0) = 0)$

then $a_{2k} = \frac{1}{k!}$ and $a_{2k+1} = 0$, $k \geq 0$

$$y_1(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k} = e^{x^2}$$

17 If $a_0 = 0$ & $a_1 = 1$, then

$$a_{2k} = 0 \quad \text{and} \quad a_{2k+1} = \frac{2^k}{(2k+1)!!} \text{ for } k \geq 0$$

$$y_2(x) = \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{(2k+1)!!}$$