Example 1 An example of series solution

Consider the equation

$$(3-x^2)y'' - 3xy' - y = 0$$

(a) Seek power series solution about $x_0 = 0$; find the recurrence relation

Solution

i) $y(x) = \sum_{n=0}^{\infty} a_n x^n$

ii) $y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \Rightarrow xy' = \sum_{n=1}^{\infty} n a_n x^n$

(note that the terms of $n=0$ and $n=1$ are actually 0)

iii) $y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$(3-x^2)y''(x) = (3-x^2)\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = 3\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$= 3\sum_{n=0} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1) a_n x^n$ (change it to $\sum_{n=2}^{\infty}$) (shift of parameter) $= 3\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) a_n x^n$

$= \sum_{n=0}^{\infty} \left(3(n+2)(n+1) a_{n+2} - n(n-1) a_n\right) x^n$

Combining all together we get $\sum_{n=0}^{\infty} \left(3(n+2)(n+1) a_{n+2} - n(n-1) a_n - 3n a_n - a_n\right) x^n = 0 \Rightarrow$

Page 2 $3(n+2)(n+1)a_{n+2} - (\underbrace{n(n-1) + 3n + 1}_{n^2+2n+1=(n+1)^2})a_n = 0$

So, $3(n+2)\cancel{(n+1)}a_{n+2} - (n+1)^{\cancel{2}} a_n = 0 \Rightarrow$ the recurrence relation is

$$\boxed{a_{n+2} = \frac{n+1}{3(n+2)} a_n} \Leftrightarrow a_n = \frac{n-1}{3n} a_{n-2}$$

(b) Find an explicit formula for coefficients a_n

$$a_n = \frac{n-1}{3n} a_{n-2} = \frac{(n-1)(n-3)}{3n\,3(n-2)} a_{n-4} = \frac{(n-1)(n-3)}{3^2 n(n-2)} a_{n-4} =$$

$$= \frac{(n-1)(n-3)(n-5)}{3^3 n(n-2)(n-4)} a_{n-6} = \begin{cases} \frac{(n-1)}{3^k n} \cdots \frac{3 \cdot 1}{4 \cdot 2} a_0 & n = 2k \Rightarrow k = \frac{n}{2} \\ \frac{(n-1)}{3^k n} \cdots \frac{4 \cdot 2}{5 \cdot 3} a_1 & n = 2k+1,\ k = \frac{n}{2} - \frac{1}{2} \end{cases}$$

Combining both

$$a_n = \frac{(n-1)!!}{3^{[\frac{n}{2}]} n!!}$$

$[\frac{n}{2}] \to$ the integer part of $\frac{n}{2}$ (the maximal integer that $\leq$ than $\frac{n}{2}$)

(c) Find a fundamental set of solutions

Let y_1 correspond to $a_0 = 1$ and $a_1 = 0 \Rightarrow a_{2k+1} = 0$

$$\boxed{y_1(x) = \sum_{k=0}^{\infty} \frac{1}{3^k} \frac{1 \cdot 3 \ldots (2k-1)}{2 \cdot 4 \ldots 2k} x^{2k}}$$

let y_2 corresponding to $a_0 = 0$, $a_1 = 1 \Rightarrow a_{2k} = 0$

$$y_2(x) = \sum_{k=0}^{\infty} \frac{1}{3^k} \frac{2 \cdot 4 \cdots 2k}{3 \cdots (2k+1)} x^{2k+1}$$

So y_1 & y_2 as above form a fundamental set of solutions

c) Write 4 first nonzero terms in the expansion of $y_1(x)$ & $y_2(x)$

$$y_1(x) = \underset{k=0}{1} + \underbrace{\frac{1}{3} \cdot \frac{1}{2}}_{k=1\ \frac{1}{6}} x^2 + \underbrace{\frac{1}{3^2} \frac{1 \cdot 3}{2 \cdot 4}}_{k=2\ \frac{1}{24}} x^4 +$$

$$+ \underbrace{\frac{1}{3^3} \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}}_{k=3\ \frac{5}{432}} x^6 + \ldots = \boxed{1 + \frac{x^2}{6} + \frac{x^4}{24} + \frac{5x^6}{432} + \ldots}$$

$$y_2(x) = \underset{k=0}{x} + \underbrace{\frac{1}{3} \frac{2}{3}}_{2/9} x^3 + \underbrace{\frac{1}{3^2} \cdot \frac{2 \cdot 4}{3 \cdot 5}}_{\frac{8}{135}} x^5 +$$

$$+ \underbrace{\frac{1}{3^3} \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}}_{\frac{16}{945}} x^7 = \boxed{x + \frac{2}{9} x^3 + \frac{8}{135} x^5 + \frac{16}{945} x^7}$$

d) What are the lower bound for the radius of convergence of this series

$$\underbrace{(3-x^2)}_{P}y'' \underbrace{-3x}_{Q}y' \underbrace{-}_{R}y = 0$$

$$\frac{Q}{P} = -\frac{3x}{3-x^2} \quad \rightarrow \text{ singular points are}$$

$$\frac{R}{P} = -\frac{1}{3-x^2} \qquad 3-x^2=0 \iff x = \pm\sqrt{3}$$

$$\Rightarrow \boxed{R_{conv} \geq \sqrt{3}}$$

Remark In fact, from the expansion of item b) you can show that $R_{conv} = \sqrt{3}$

For example for $y_1(x)$

$$y_1(x) = \sum_{k=0}^{\infty} \frac{1}{3^k} \underbrace{\frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots 2k}}_{b_k = a_{2k}} (x^2)^k$$

$\Rightarrow$ Page 2 $a_{2k+2} = \frac{2k+1}{3(2k+2)} a_{2k}$

$$\frac{b_k}{b_{k+1}} = \frac{a_{2k}}{a_{2k+2}} = \frac{3(2k+2)}{2k+1} \xrightarrow[k\to\infty]{} 3 \Rightarrow$$

$$x^2 < 3 \Rightarrow \boxed{|x| < \sqrt{3}}$$

Example 2 (Phase portrait of planar linear systems)

1. For given linear nonhom system classify critical point, determine its stability property and sketch the phase portrait

1a) $\begin{cases} x_1' = -6x_1 + 3x_2 \\ x_2' = -x_1 - 2x_2 - 5 \end{cases}$

Solution

Find the critical point

$\begin{cases} -6x_1 + 3x_2 = 0 \\ -x_1 - 2x_2 = 5 \end{cases} \Leftrightarrow \begin{cases} 2x_1 - x_2 = 0 & (Eq\ 1) \\ -x_1 - 2x_2 = 5 & (Eq\ 2) \end{cases}$

$(Eq\ 1) + 2Eq\ 2: \ -5x_2 = 5 \Rightarrow x_2 = -1 \overset{(Eq\ 1)}{\Rightarrow} -2 - x_2 = 0 \Rightarrow x_2 = -2$

Crit. point is $(-1, -2)$

$A = \begin{pmatrix} -6 & 3 \\ -1 & -2 \end{pmatrix}$ $\quad$ $\operatorname{tr} A = -8$, $\det A = 15$

$\lambda^2 + 8\lambda + 15 = 0$

$D = 64 - 60 = 4$

$\lambda_1 = \frac{-8+2}{2} = -3$

$\lambda_2 = \frac{-8-2}{2} = -5$

$\rightarrow$ real negative $\Rightarrow$ **nodal sink** $\rightarrow$ **asympt. stable**

Find eigenlines: i) $\lambda_1 = -3$ $\quad (A - \lambda_1 I)v = (A + 3I)v = \begin{pmatrix} -3 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$

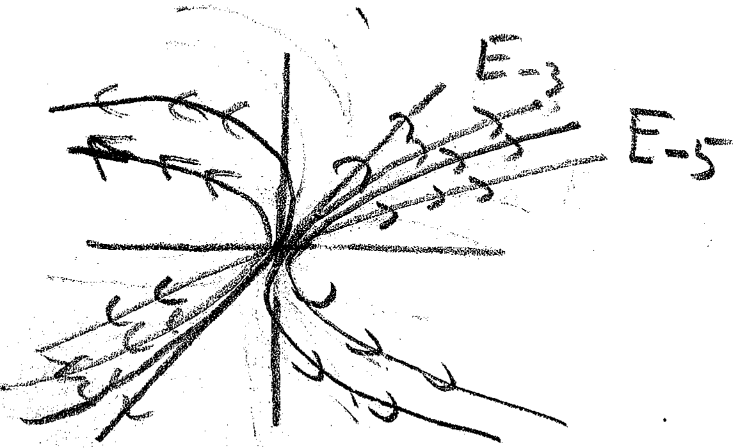
$\Leftrightarrow -v_1 + v_2 = 0 \Rightarrow v_1 = v_2 \Rightarrow E_{-3}$ is generated by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

ii) $\lambda_2 = -5$

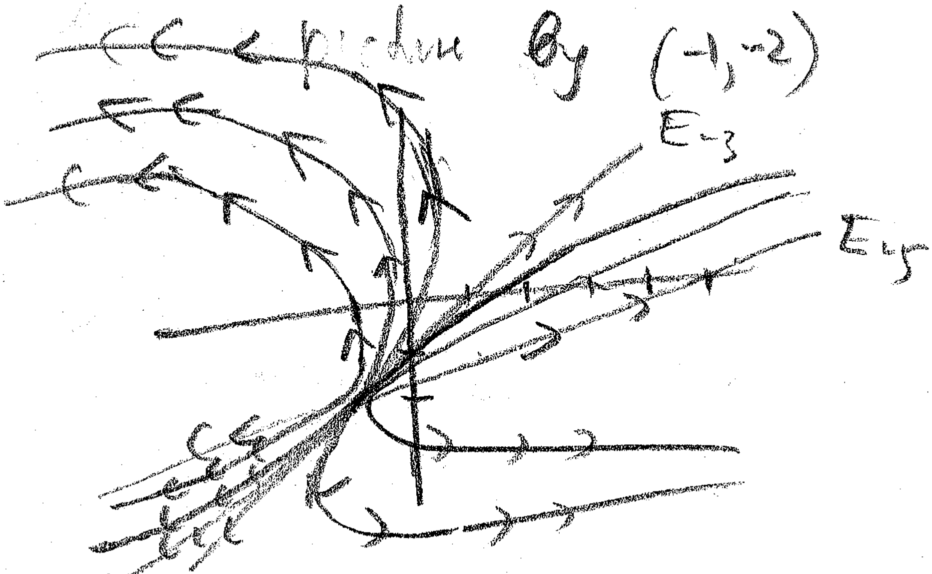
$$(A-(-5)I)v = (A+5I)v = \begin{pmatrix} -1 & 3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$-v_1 + 3v_2 = 0 \Rightarrow v_1 = 3v_2 \Rightarrow$ E_{-5} is generated by $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$|\lambda_2| > |\lambda_1| \Rightarrow$ trajectories go out of the origin with velocities close to E_{-5} while as the trajectory approaches infinity its tangent lines becomes almost parallel to E_{-5}



now shift the picture by $(-1,-2)$



B) $\begin{cases} x_1' = 3x_1 - x_2 - 5 \\ x_2' = x_1 + 5x_2 - 7 \end{cases}$

Solutions

- Critical points

$\begin{cases} 3x_1 - x_2 = 5 & \text{Eq 1} \\ x_1 + 5x_2 = 7 & \text{Eq 2} \end{cases}$

5(Eq 1) + (Eq 2): $16x_1 = 32 \Rightarrow x_1 = 2 \Rightarrow$

$x_2 = 3x_1 - 5 = 6 - 5 = 1$

$\Rightarrow (2,1)$ is the only critical points

- $A = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix}$ $\quad$ $\text{tr} A = 8$, $\det A = 16$

$\Rightarrow$ char. polynomial is

$\lambda^2 - 8\lambda + 16 = 0 \Leftrightarrow (\lambda - 4)^2 = 0$

$\lambda_{1,2} = 4 > 0 \to$ alg. mult. is 2

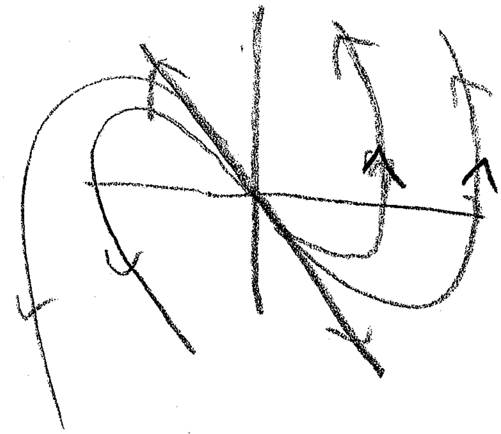
But $A \neq 4I \Rightarrow$ geom. mult =

$\Rightarrow$ **Improper nodal source** $\to$ **unstable**

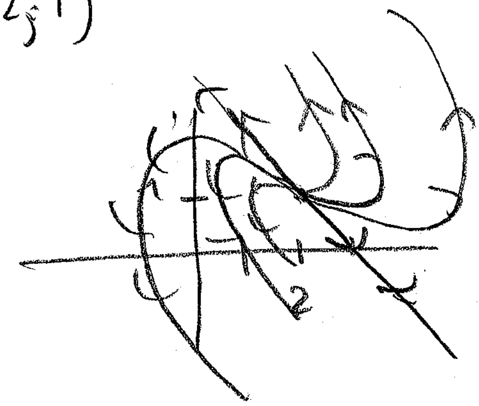
- Find the eigen line: $(A - 4I)v = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$v_1 + v_2 = 0 \Rightarrow E_4$ is generated by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

• $a_{21} = 1 > 0 \Longrightarrow$ the part of trajectory far from the origin moves counter-clockwise

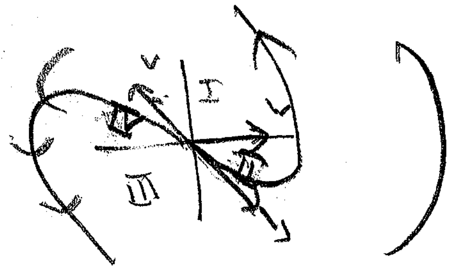


shift it by $(2;1)$ ⟶



(by method using v and w

if $w = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow v = (A - \lambda I)\, w = \begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$



)

(c) $\begin{cases} x_1' = -5x_1 + 17x_2 - 2 \\ x_2' = -2x_1 + x_2 + 5 \end{cases}$

Solution

. Critical points $\begin{cases} -5x_1 + 17x_2 = 2 & (Eq\ 1) \\ -2x_1 + x_2 = -5 & (Eq\ 2) \end{cases}$

$(Eq\ 1) - 17(Eq\ 2)$: $(-5 + 34)x_1 = 2 + 85 \Rightarrow$

$29x_1 = 87 \Rightarrow x_1 = 3 \overset{(Eq\ 2)}{\Rightarrow} -6 - x_2 = -5 \Rightarrow$

$x_2 = 1$

So $(3,1)$ is the only critical point

- $A = \begin{pmatrix} -5 & 17 \\ -2 & 1 \end{pmatrix} \Rightarrow \operatorname{tr} A = -4$

$\det A = -5 + 34 = 29$

⇓

Char. eq. $\lambda^2 + 4\lambda + 29 = 0 \Leftrightarrow \lambda^2 + 4\lambda + 4 + 25 = (\lambda+2)^2 + 5^2 = 0$ (complete squares)

$\Rightarrow \lambda_{1,2} := -2 \pm 5i$

or $D = 16 - 116 = -100$

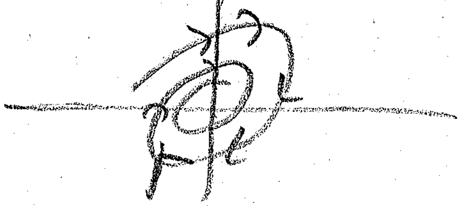
$\lambda_{1,2} = \frac{-4 \pm 10i}{2} = -2 \pm 5i$

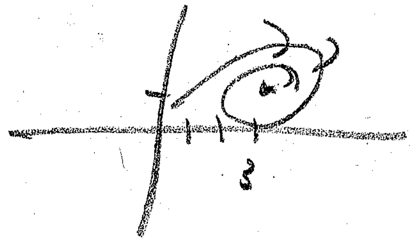
λ are complex and

$Re\, \lambda_{1,2} := -2 < 0 \Rightarrow$ spiral sink $\Rightarrow$ asymptotically stable

- direction $a_{21} < 0 \Rightarrow$ clockwise

shift in 3, 1





Page 9) Check using calculation of the eigenvector

$v = a + ib$

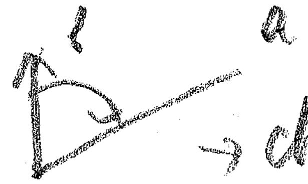
$\lambda = -2+5i$

$(A - (-2+5i)I)v = (A + (2-5i)I)v =$

$$= \begin{pmatrix} -3-5i & 17 \\ -2 & 3+5i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$-(3+5i)v_1 + 17v_2 = 0$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 17 \\ 3+5i \end{pmatrix} = \underbrace{\begin{pmatrix} 17 \\ 3 \end{pmatrix}}_{a} + i\underbrace{\begin{pmatrix} 0 \\ 5 \end{pmatrix}}_{b}$$



→ clockwise.

Example 3 (how to analyze non-linear systems using linearization) (Page 7)

$$\begin{cases} \frac{dx}{dt} = x + x^2 - y^2 \\ \frac{dy}{dt} = y - xy \end{cases}$$

(a) Find critical points

$$\begin{cases} x + x^2 - y^2 = 0 & (Eq\,1) \\ y - xy = 0 & (Eq\,2) \end{cases}$$

Factor (Eq 2) : $y(1-x) = 0$

2 options

1) $y = 0 \Rightarrow$ (substitute to equation 1)

$x + x^2 = 0 \Rightarrow x(x+1) = 0 \Rightarrow x = 0$ or $x = -1$

$\Rightarrow (0,0)$ and $(-1,0)$

2) $x = 1 \Rightarrow$ substitute to (Eq 1) $\Rightarrow$

$2 - y^2 = 0 \Rightarrow y = \pm\sqrt{2}$

$(1, \sqrt{2})$ and $(1, -\sqrt{2})$

4 critical points : $(0,0)$, $(-1,0)$, $(1,\sqrt{2})$, $(1,-\sqrt{2})$

(b) Find the linearization at each critical point, and classify each critical point and find its stability property.

First calculate the Jacobi matrix

$f(x,y) = x + x^2 - y^2$

$g(x,y) = y - xy$

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 1+2x & -2y \\ -y & 1-x \end{pmatrix}$$

1) $(x_0, y_0) = (0,0)$ $\quad J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\lambda_1 = \lambda_2 = 2 \rightarrow$ linearization $\rightarrow$ proper nodal source

$\rightarrow$ original system $\rightarrow$ nodal source

(for original system all adjectives like proper / improper should be removed)

2) $(x_0, y_0) = (-1, 0)$

$$J(-1,0) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

$\Rightarrow \lambda_1 = -1, \ \lambda_2 = 2 \rightarrow$ real, opposite sign $\rightarrow$ saddle point

3) $(x_0, y_0) = (1, \sqrt{2})$

$$J(1, \sqrt{2}) = \begin{pmatrix} 3 & -2\sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix}$$

$\det J = -4 < 0$

$\det J < 0 \Rightarrow$ saddle point

4) $(x_0, y_0) = (1, -\sqrt{2})$

$$J(1,-\sqrt{2}) = \begin{pmatrix} 3 & 2\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$$

$\det J(1,-\sqrt{2}) = -4 < 0 \Rightarrow$ saddle point

(c) Sketch the phase portrait

Let us find the eigenlines of linearization of the saddles (they will be tangent to separatrices at critical points

i) $(x_0, y_0) = (-1, 0)$ $J(-1,0) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

eigenlines are axis (x-axis is stable, y-axis is unstable)

ii) $(x_0, y_0) = (1, \sqrt{2})$ $J(1,\sqrt{2}) = \begin{pmatrix} 3 & -2\sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix}$

$\text{tr } J = 3$ $\det = -4$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$D = 9 + 16 = 25$$

$$\lambda_1 = \frac{3-5}{2} = -1$$

$$\lambda_2 = \frac{3+5}{2} = 4$$

· Eigenline of $\lambda=-1$

$$(J+I)v = \begin{pmatrix} 4 & -2\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} v = 0 \Leftrightarrow -\sqrt{2}v_1 + v_2 = 0$$

$v_1 = 1,\ v_2 = \sqrt{2}$

The eigenline is generated by $\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$

Eigenline of $\lambda = 4$

$$(J-4I)v = \begin{pmatrix} -1 & -2\sqrt{2} \\ -\sqrt{2} & -4 \end{pmatrix} v = 0 \Leftrightarrow v_1 + 2\sqrt{2}v_2 = 0$$

$\Rightarrow v_1 = -2\sqrt{2}v_2 \Rightarrow$ The eigenline is generated by

$\begin{pmatrix} -2\sqrt{2} \\ 1 \end{pmatrix}$

iii) $(x_0, y_0) = (1, -\sqrt{2})$ $\quad J = \begin{pmatrix} 3 & 2\sqrt{2} \\ 2\sqrt{2} & 0 \end{pmatrix}$

The same eigenvalues $\lambda=-1$, $\lambda=4$

(because the char. pol. is the same)

For $\lambda=-1$ the eigenline is generated by $\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$

For $\lambda=4$ the eigenline is generated by $\begin{pmatrix} 2\sqrt{2} \\ 1 \end{pmatrix}$

For the sketch see another file!!!