12. System of homogeneous linear equations with constant coefficients: the role of eigenvalues and eigenvetors, the case of distinct eigenvalues(sec 7.3 and 7.5)

1. A number λ is called an **eigenvalue** of matrix A if there exists a **nonzero** vector $\mathbf{v}$ such that
$$A\mathbf{v} = \lambda \mathbf{v},$$
and $\mathbf{v}$ is called an **eigenvector** corresponding to the eigenvalue λ.

2. Example (corresponds to uncoupled systems). If A is a *diagonal matrix*,
$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$
then the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues and the vectors
$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \quad \mathbf{v_n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$
are the corresponding eigenvectors.

3. Relation of diagonal matrices to uncoupled systems:

4. **Fundamental Proposition.** *If λ is an eigenvalue of matrix A and $\mathbf{v}$ is an eigenvector corresponding to this eigenvalue then*
$$\mathbf{X}(t) = e^{\lambda t}\mathbf{v}$$
is a solution of the system $\mathbf{X}' = A\mathbf{X}$, i.e solution of the homogeneous linear system with constant coefficients.

5. Geometric interpretation of Fundamental Proposition.

6. Eigenvalue are solutions of the following **characteristic equation (roots of the following characteristic polynomial):**
$$\det(A - \lambda I) = 0.$$
What is the degree of this polynomial?

7. **Trace of an $n \times n$ matrix** A is the sum of it diagonal elements, denoted by trace(A) or tr(A):

$$\mathrm{trace}(A) = a_{11} + a_{22} + \dots a_{nn}.$$

8. Show that the characteristic equation in the case $n = 2$ can be found as

$$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A) = 0.$$

9. Consequently to find eigenvalues of an 2×2 matric we need to solve a quadratic equation (and more generally, to find eigenvalues of $n \times n$ matrix we need to find roots of a polynomial of degree n).

10. `Fact from Algebra:` The quadratic equation $a\lambda^2 + b\lambda + c = 0$ has roots

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

which fall into one of 3 cases:

- two distinct real roots $\lambda_1 \neq \lambda_2$ (in this case $D = b^2 - 4ac > 0$) [corresponds to part of section 7.5 and can be applied to section 3.1
- two complex conjugate roots $\lambda_1 = \overline{\lambda_2}$ (in this case $D = b^2 - 4ac < 0$) [corresponds to a part of section 7.6 and can be applied to section 3.3]
- two equal real roots $\lambda_1 = \lambda_2$ (in this case $D = b^2 - 4ac = 0$) [[corresponds to a part of section 7.8 and can be applied to section 3.4]

Real Distinct Eigenvalues

11. *FACT from Linear algebra: If A has* <u>*distinct*</u> *eigenvalues $\lambda_1, \dots, \lambda_n$ and $\mathbf{v}^1, \dots, \mathbf{v}^n$ are the corresponding eigenvectors, then* $\det(v^1, \dots, v^n)$ *(i.e. the determinant of the matrix with jth column equal to $\mathbf{v}^j$) does not vanish or, equivalently the collection of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ form a basis of $\mathbb{R}^n$*

   As a consequence, if If A has <u>distinct real</u> eigenvalues $\lambda_1, \dots, \lambda_n$ with eigenvectors $\mathbf{v}^1, \dots, \mathbf{v}^n$, then

$$\left\{e^{\lambda_1 t}\mathbf{v}^1, \dots, e^{\lambda_n t}\mathbf{v}^n\right\}$$

   is a fundamental set of solutions and the general solution is

$$\mathbf{X}(t) = C_1 e^{\lambda_1 t}\mathbf{v}^1 + \dots + C_n e^{\lambda_n t}\mathbf{v}^n. \tag{1}$$

   REMARK If the eigenvalues are distinct but some of them are complex, the formula (1) gives the general complex-valued solutions, so we need to make additional work to get the general real-valued solutions (will be discussed next week and corresponds to section 7.6 and 3.3, we also will make a thorough review of complex numbers there).

12. EXAMPLE. Consider the following system of ODEs:

$$\begin{aligned} x_1' &= -2x_1 + x_2 \\ x_2' &= 2x_1 - 3x_2 \end{aligned} \tag{2}$$

(a) Find general solution of (2).

(b) Find solution of (2) subject to the initial condition $X(0) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

(c) What is behavior of the solution as $t \to +\infty$?

13. Applications to second and higher order linear homogeneous equations (sections 3.2, 3.1 for second order, 4.1, 4.2 for higher order, the latter can be skipped)

1. Consider a linear homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0 \tag{3}$$

with coefficients p and q being continuous in an interval I. Then, as was already discussed in section 9 item 7 of the notes, this equation can be transformed to the following system of first order equations, by setting $x_1(t) := y(t)$, $x_2(t) = y'(t)$:

$$\begin{cases} x_1' = x_2 \\ x_2' = -q(t)x_1 - p(t)x_2. \end{cases} \tag{4}$$

so that a function $y(t)$ is a solution of (3) if and only of the vector function

$$\mathbf{X}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

is a solution of (4).

Then as a consequence of general theory for systems (section 10, items 16-18, and section 11, items 8-12) we get

2. **Superposition Principle for second (and also higher) order equations** reads: If $y_1(t)$ and $y_2(t)$ are two solutions of (3), then

$$y(t) = C_1 y_1(t) + C_2 y_2(t). \tag{5}$$

is a solution of (3).

3. **Wronskian of two solutions** Take two solutions $y_1(t)$ and $y_2(t)$ of (3). Then then

$$\mathbf{X}_1(t) = \begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix}, \mathbf{X}_2(t) = \begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix}$$

are two solutions of (4). Therefore the 2×2 matrix $\Psi(t)$ formed from this two solutions is

$$\Psi(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \tag{6}$$

DEFINITION 1. *The determinant of the matrix $\Psi(t)$ is called* **WRONSKIAN** *of the functions $y_1(t)$ and $y_2(t)$ and it is denoted by $W(y_1, y_2)(t)$:*

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

4. As a consequence of Theorem 2 of section 10 we get

THEOREM 2. *Let $y_1(t)$ and y_2 are solutions of (3). The general solution of (3) is given by $y(t) = C_1y_1(t) + C_2y_2(t)$ if and only if the Wronskian $W(y_1, y_2)(t_0) \neq 0$ for some time moment t_0.*

5. As discussed in the very first lecture $\cos t$ and $\sin t$ are solutions of $y'' + y = 0$. Using the developed theory, justify that $y(t) = C_1 \cos t + C_2 \sin t$ is the general solution of this equation

6. **Generalization for equation of order n (chapter 4)** What is the Wronskian for n solutions of an equation of nth order and the analog of Theorem (2) in this case?

The case of linear homogeneous equations of second order: characteristic equation and general solution in the the case of real distinct roots (sec. 3.1)

7. Consider

$$ay'' + by' + cy = 0 \tag{7}$$

with constant real coefficients a, b, and c, $a \neq 0$. The corresponding system of first order equation is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\tfrac{c}{a}x_1 - \tfrac{b}{a}x_2. \end{aligned} \tag{8}$$

8. Show that the characteristic equation for the eigenvalues of the matrix of the system (8) is equivalent to

$$a\lambda^2 + b\lambda + c = 0. \tag{9}$$

Note that the characteristic equation (9) can be determined from the original second differential equation (7) simply by replacing $y^{(k)}$ with λ^k (you relate to y itself as to the derivative of y of order 0).

The case of two distinct real roots λ_1 and λ_2 of (9) $\Leftrightarrow$ distinct real eigenvalues of the matrix of the corresponding system (8). Therefore the general solution of (7) is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2} t. \tag{10}$$

EXPLANATION

9. More elementary derivation of (9) and (10) without using the notions of eigenvalues and eigenvectors: the nature of the equation (7) suggests that it may have solutions of the form$y = e^{\lambda t}$. Plug it to (7):

10. EXAMPLE. Consider

$$3y'' - y' - 2y = 0.$$

    (a) Find general solution.
    (b) Find solution satisfying the following initial conditions: $y(0) = \alpha, \quad y'(0) = 1$, where α is a real parameter.
    (c) Find all α so that the solution of the corresponding IVP approaches 0 as $t \to +\infty$.

11. How to generalize this theory to linear equation with constant coefficients of order n?