25: Power Series, Taylor Series and Analytic Functions (section 5.1)

DEFINITION 1. *A power series about $x = x_0$ (or centered at $x = x_0$), or just* **power series**, *is any series that can be written in the form*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where x_0 and a_n are numbers.

DEFINITION 2. *A power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is said to converge at a point x if the limit $\lim_{m\to\infty} \sum_{n=0}^{m} a_n(x - x_0)^n$ exists and finite.*

REMARK 3. *A power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ always converges at $x = x_0$.*

EXAMPLE 4. *For what x the power series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$ converges ?*

$\sum_{n=0}^{m} x^n =$ $1 + x + \ldots + x^m = \frac{1 - x^{m+1}}{1-x}$

If $|x| < 1$ $\lim_{m\to\infty} x^m = 0 \Rightarrow \sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$

If $|x| > 1$ $\lim_{m\to\infty} x^m = \infty \Rightarrow \sum_{n=0}^{\infty} x^n$ diverges

Absolute Convergence: The series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is said to converge absolutely at x if $\sum_{n=0}^{\infty} |a_n|\,|x - x_0|^n$ converges.

If a series converges absolutely then it converges (but in general not vice versa).

EXAMPLE 5. *The series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges at $x = -1$, but it doesn't converges absolutely there:*

$$1 - \frac{1}{2} + \frac{1}{3} - \ldots = \ln 2$$

but the series of absolute values is the so-called harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

and it is divergent.

Fact: If the series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges absolutely at $x = x_1$ then it converges absolutely for all x such that $|x - x_0| < |x_1 - x_0|$

This immediately implies the following:

THEOREM 6. *For a given power series* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *there are only 3 possibilities:*

1. *The series converges only for* $x = x_0$.
2. *The series converges for all* x.
3. *There is* $R > 0$ *such that the series converges if* $|x-x_0| < R$ *and diverges if* $|x-x_0| > R$. *We call such* R *the* **radius of convergence.**

REMARK 7. In case 1 of the theorem we say that $R = 0$ and in case 2 we say that $R = \infty$

EXAMPLE 8. *What is the radius of convergence of the geometric power series* $\sum_{n=0}^{\infty} x^n$*?*

According to example 4 $R=1$

How to find Radius of convergence: If $a_n \neq 0$ for any n and $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ exists, then

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

More generally,

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}},$$

the Cauchy-Hadamard formula.

EXAMPLE 9. *Find the radius of convergence of the power series* $\sum_{n=0}^{\infty} \frac{n^2}{3^n}(x+1)^n$.

$$\frac{a_n}{a_{n+1}} = \frac{\frac{n^2}{3^n}}{\frac{(n+1)^2}{3^{n+1}}} = 3\frac{n^2}{(n+1)^2} \xrightarrow[n\to\infty]{} 3 \Rightarrow \boxed{R=3}$$

The Taylor series for $f(x)$ about $x = x_0$

Assume that f has derivatives of any order at $x = x_0$. Then for any m

$$f(x) = \sum_{n=0}^{m} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(m+1)}(c)}{(m+1)!}(x-x_0)^{m+1}$$

where c is a number between x and x_0. The remainder converges to zero at least as fast as $(x-x_0)^{m+1}$ when $x \to x_0$.

Formally we can consider the following power series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

the *Taylor series of the function* f *about* x_0.

The Taylor series may converge and may not converge in a neighborhood of x_0 and even if it converges for any x close to x_0 it may not converge to $f(x)$.

DEFINITION 10. *The function* f *is called* **analytic** *at the point* x_0 *if there exist a power series*

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n$$

such that

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

for all x *sufficiently close to* x_0.

In this case, one can show that a_n must be equal to $\dfrac{f^{(n)}(x_0)}{n!}$. This implies:
The function f *is analytic at the point* x_0 *if its Taylor series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

converges to $f(x)$ *for all* x *sufficiently closed to* x_0.

EXAMPLE 11. *From Example 3 it follows that* $f(x) = \dfrac{1}{1-x}$ *is analytic at* 0. *In general, it is analytic at any* $x_0 \neq 1$

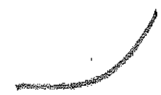
EXAMPLE 12. *More generally, any rational function* $f(x) = \dfrac{Q(x)}{P(x)}$, *where* $P(x)$ *and* $Q(x)$ *are polynomials without common linear factors (the latter can be always assumed, because the common factors can be canceled) is analytic at all points except zeros of the denominator* $P(x)$

EXAMPLE 13. *The functions* e^x, $\sin x$ *and* $\cos x$ *are analytic at any* x. *Here are their Taylor series at* 0:

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} &&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} &&= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots
\end{aligned}$$

REMARK 14. *Not any function having derivatives of any order at any point is analytic. For example, take*

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \le 0 \end{cases}.$$



$\lim\limits_{h\to 0+} \frac{e^{-\frac{1}{h}} - 0}{h} = \lim\limits_{h\to 0+} \frac{1}{h} e^{-\frac{1}{h}} = \lim\limits_{x\to +\infty} x e^{-x} = 0 \Rightarrow$ $x = \frac{1}{h}$ $f'(0) = 0$ Similarly $f^{(n)}(0) = 0$ for any n

The Taylor series of f at 0 is identically $= 0$

Term by term differentiation If f is analytic at a point x_0, i.e $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ for x sufficiently close to x_0, then $f'(x)$ is also analytic at x_0 and

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-x_0)^n$$

In other words, the derivative of a (convergent) power series is obtained by term by term differentiation of the series.

EXAMPLE 15. *What is the Taylor expansion of $f''(x)$?*

$$f''(x) = \left(\sum_{n=0}^{\infty} (n+1) a_{n+1} (x-x_0)^n\right)' = \sum_{n=1}^{\infty} (n+1) n a_{n+1} (x-x_0)^{n-1} =$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-x_0)^n.$$