26: Series of Solutions near an Ordinary Point (sections 5.2 and 5.3)

We consider differential equations of the type

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

DEFINITION 1. *A point x_0 is called an* **ordinary point** *of differential equation* (1) *if the functions $p(x) := \dfrac{Q(x)}{P(x)}$ and $q(x) := \dfrac{R(x)}{P(x)}$ are analytic at x_0, maybe after defining $p(x_0)$ and $q(x_0)$ appropriately (i.e. by continuity). Otherwise, the point x_0 is called a* **singular point** *of differential equation* (1).

EXAMPLE 2. *If the functions $P(x)$, $Q(x)$, and $R(x)$ are analytic and $P(x_0) \neq 0$, then x_0 is an ordinary point of* (1).

$$\frac{Q(x)}{P(x)} = \frac{Q(x)}{P(x_0)+P'(x_0)(x-x_0)+\dots} = \frac{Q(x)}{P(x_0)}\,\frac{1}{1+\underbrace{\frac{P'(x_0)}{P(x_0)}(x-x_0)+\dots}_{\rho}} = \frac{Q(x)}{P(x_0)}(1-\rho+\rho^2-\dots)$$

REMARK 3. *In general, if $P(x_0) = 0$ it does not mean that x_0 is a singular point of* (1), *because it might be that also $Q(x_0) = 0$ and $R(x_0) = 0$ and the functions $\dfrac{Q(x)}{P(x)}$, $\dfrac{R(x)}{P(x)}$ might be defined at x_0 by continuity (namely $\lim\limits_{x\to x_0} \dfrac{Q(x)}{P(x)}$ and $\lim\limits_{x\to x_0} \dfrac{R(x)}{P(x)}$ may exist and be finite).*

EXAMPLE 4. *Given*

$$(1-x^2)y'' + (x^2+x-2)y'' + (x^3-1)y = 0$$

a) is $x_0 = 1$ ordinary or singular?

$$\frac{x^2+x-2}{1-x^2} = \frac{(x-1)(x+2)}{(1-x)(1+x)} = -\frac{x+2}{x+1} \rightarrow \text{ordinary by example 2}$$

$$\frac{x^3-1}{1-x^2} = \frac{(x-1)(x^2+x+1)}{(1-x)(1+x)} = -\frac{x^2+x+1}{x+1}$$

b) is $x_0 = -1$ ordinary or singular?

singular, because

$$\frac{Q(x)}{P(x)} \xrightarrow[x\to -1]{} +\infty$$

EXAMPLE 5. *Given*

$$\sin^2 x\, y'' + x^2 y' + (1-\cos x)y = 0$$

a) is $x_0 = 0$ ordinary or singular?

$\lim_{x\to 0} \frac{x^2}{\sin^2 x} = 1 \to$ finite

$\lim_{x\to 0} \frac{1-\cos x}{\sin^2 x} = \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{\underbrace{\sin^2 x}_{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}} = \frac{1}{2} \to$ finite $\Rightarrow$ ordinary

b) is $x_0 = 2\pi$ ordinary or singular?

$\lim_{x\to 2\pi} \frac{x^2}{\sin^2 x} \to \infty \to$ singular $\Rightarrow$ singular

THEOREM 6. *1. If x_0 is an ordinary point of differential equation (1), then any solution $y(x)$ of (1) is analytic at $x = x_0$, i.e can be found as a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$.*

2. The radius of convergence of this series is at least as large as the minimum of the radii of convergence of the Taylor series at x_0 of functions $p(x) := \frac{Q(x)}{P(x)}$ and $q(x) := \frac{R(x)}{P(x)}$.

REMARK 7. *The coefficients a_n of the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ with $n \geq 2$ are uniquely determined by the first two coefficients a_0 and a_1. Moreover, a_n are expressed linearly in terms of a_0 and a_1 so that*

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where $y_1(x)$ is the solution satisfying the initial conditions $y_1(x_0) = 1$, $y_1'(x_0) = 0$ and $y_2(x)$ is the solution satisfying the initial conditions $y_2(x_0) = 0$, $y_2'(x_0) = 1$.

The idea of the proof: Substituting the series into equation and comparing coefficient one get a recurrence relation expressing a_n in terms of the previous coefficients and they are uniquely determined by a_0 & a_1

EXAMPLE 8. *Given differential equation*

$$xy'' + y' + xy = 0.$$

a) Seek power series solutions of this equation about $x_0 = 1$: find the recurrence relation for coefficients of the power series about $x_0 = 1$ representing a solution (in general, a recurrence relation is a relation expressing the nth coefficients a_n in terms of some previous ones).

$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ shift in index

$x y(x) = ((x-1)+1) \sum_{n=0}^{\infty} a_n (x-1)^n = \sum_{n=0}^{\infty} a_n (x-1)^n +$

$+ \underbrace{\sum_{n=0}^{\infty} a_n (x-1)^{n+1}}_{\sum_{n=1}^{\infty} a_{n-1}(x-1)^n} = a_0 + \sum_{n=1}^{\infty} (a_n + a_{n-1})(x-1)^n$

$y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n =$

$= a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} (x-1)^n$

$x y''(x) = ((x-1)+1) \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} +$

$+ \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{n=1}^{\infty} (n+1) n a_{n+1} (x-1)^n +$

$+ \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n = 2a_2 + \sum_{n=1}^{\infty} \left((n+1) n a_{n+1} + (n+2)(n+1) a_{n+2}\right)(x-1)^n$

$\Rightarrow xy'' + y' + xy = a_0 + a_1 + 2a_2 + \sum_{n=1}^{\infty} \left(a_{n-1} + a_n + (\underbrace{(n+1)n + (n+1)}_{(n+1)^2}) a_{n+1} + \right.$

b) Find the first five terms in the power expansion about $x_0 = 1$ of the solution of the equation (1) satisfying initial conditions $y(1) = 3$, $y'(1) = 1$.

Previous item is continued:

$$+ (n+1)(n+2))a_{n+2}(x-x_0)^n = a_0 + a_1 + 2a_2 + \sum_{n=1}^{\infty} (a_{n-1} + a_n + (n+1)^2 a_{n+1} + (n+1)(n+2))a_{n+2}(x-x_0)^n \Rightarrow$$

comparing coefficients

$$a_0 + a_1 + 2a_2 = 0 \Rightarrow \boxed{a_2 = -\frac{1}{2}(a_0 + a_1)}$$

$$a_{n-1} + a_n + (n+1)^2 a_{n+1} + (n+1)(n+2) a_{n+2} = 0 \Rightarrow$$

for $n \geq 1$.

The recurrence relation is

together

$$\boxed{a_{n+2} = -\frac{(n+1)^2 a_{n+1} + a_n + a_{n-1}}{(n+1)(n+2)}}, \quad n \geq 1$$

(b) $a_0 = 3$, $a_1 = 1$ $\qquad a_2 = -\frac{1}{2}(3+1) = -2$

$n=1$: $\quad a_3 = -\frac{4a_2 + a_1 + a_0}{2 \cdot 3} = -\frac{-8+3+1}{6} = \boxed{\frac{2}{3}}$

$n=2$: $\quad a_4 = -\frac{9a_3 + a_2 + a_1}{3 \cdot 4} = -\frac{9 \cdot \frac{2}{3} - 2 + 1}{12} = \boxed{-\frac{5}{12}}$

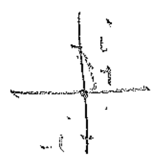
Radius of convergence as the minimal distance to singularities.

There is a more elegant way to find the radius of convergence of the Taylor series at x_0 of the analytic function f rather than calculating the coefficients of this Taylor series (i.e. derivatives of any order of f at x_0). For this we pass to the complex plane.

Assume for simplicity that $f(x) = \frac{Q}{P}$, where Q and P are polynomials and all common linear factors (in general with complex coefficients) of Q and P are canceled. Then f is analytic at x_0 if and only if $P(x_0) \neq 0$ and **the radius of convergence of the Taylor series about x_0 is equal to the distance to the nearest complex zero of P.**

EXAMPLE 9. *Given $f(x) = \dfrac{1}{1+x^2}$ what is the radius of convergence of the Taylor series of f*

a) around $x_0 = 0$ $\qquad 1+x^2=0 \Rightarrow x=\pm i$



the from 0 to $\pm i$ is 1

b) around $x_0 = 1$

$\boxed{\sqrt{2}}$

c) around $x_0 = 2$

$\boxed{\sqrt{5}}$

REMARK 10. *Note that the function $f(x) = \dfrac{1}{1+x^2}$ of the previous example is defined for all real x, so we see the "problem" (non-convergence of Taylor series for $|x| > 1$) only when passing to the complex plane. This is another example when the use of complex numbers is very natural and helpful.*

EXAMPLE 11. *Determine a lower bound for the radius of convergence of series solutions about each given point of the following equation:*

a) $xy'' + y' + xy = 0$ about $x_0 = 1$ (as in Example 8)

$$\frac{Q(x)}{P(x)} = \frac{1}{x}, \quad \frac{R(x)}{P(x)} = 1 \Rightarrow$$

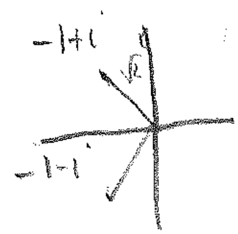
$x = 0$ is the only singular point, the distance from $x_0 = 1$ to it is $\boxed{1}$

b) $(x^2 + 2x + 2)y'' + xy' + 4y = 0$ about

i) $x_0 = 0$

$$\frac{Q(x)}{P(x)} = \frac{x}{x^2+2x+2} \qquad \frac{R(x)}{P(x)} = \frac{4}{x^2+2x+2}$$

$x^2+2x+2=0 \iff (x+1)^2+1=0$, $x = -1 \pm i$

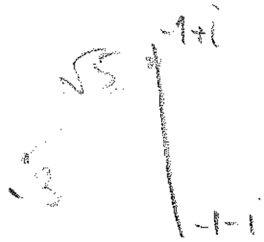


minimal distance = $\boxed{\sqrt{2}}$

ii) $x_0 = -1$

$\Rightarrow$ minimal distance is 1

iii) $x_0 = -3$



$$|-1-i-(-3)| = |2-i| = \sqrt{2^2+1} = \boxed{\sqrt{5}}$$