

## 26: Series of Solutions near an Ordinary Point (sections 5.2 and 5.3)

We consider differential equations of the type

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (1)$$

DEFINITION 1. A point  $x_0$  is called an **ordinary point** of differential equation (1) if the functions  $p(x) := \frac{Q(x)}{P(x)}$  and  $q(x) := \frac{R(x)}{P(x)}$  are analytic at  $x_0$ , maybe after defining  $p(x_0)$  and  $q(x_0)$  appropriately (i.e. by continuity). Otherwise, the point  $x_0$  is called a **singular point** of differential equation (1).

EXAMPLE 2. If the functions  $P(x)$ ,  $Q(x)$ , and  $R(x)$  are analytic and  $P(x_0) \neq 0$ , then  $x_0$  is an ordinary point of (1).

REMARK 3. In general, if  $P(x_0) = 0$  it does not mean that  $x_0$  is a singular point of (1), because it might be that also  $Q(x_0) = 0$  and  $R(x_0) = 0$  and the functions  $\frac{Q(x)}{P(x)}$ ,  $\frac{R(x)}{P(x)}$  might be defined at  $x_0$  by continuity (namely  $\lim_{x \rightarrow x_0} \frac{Q(x)}{P(x)}$  and  $\lim_{x \rightarrow x_0} \frac{R(x)}{P(x)}$  may exist and be finite).

EXAMPLE 4. Given

$$(1 - x^2)y'' + (x^2 + x - 2)y' + (x^3 - 1)y = 0$$

a) is  $x_0 = 1$  ordinary or singular?

b) is  $x_0 = -1$  ordinary or singular?

EXAMPLE 5. *Given*

$$\sin^2 x y'' + x^2 y' + (1 - \cos x)y = 0$$

a) is  $x_0 = 0$  ordinary or singular?

b) is  $x_0 = 2\pi$  ordinary or singular?

THEOREM 6. 1. *If  $x_0$  is an ordinary point of differential equation (1), then any solution  $y(x)$  of (1) is analytic at  $x = x_0$ , i.e can be found as a power series  $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ .*

2. *The radius of convergence of this series is at least as large as the minimum of the radii of convergence of the Taylor series at  $x_0$  of functions  $p(x) := \frac{Q(x)}{P(x)}$  and  $q(x) := \frac{R(x)}{P(x)}$ .*

REMARK 7. *The coefficients  $a_n$  of the series  $\sum_{n=0}^{\infty} a_n(x - x_0)^n$  with  $n \geq 2$  are uniquely determined by the first two coefficients  $a_0$  and  $a_1$ . Moreover,  $a_n$  are expressed linearly in terms of  $a_0$  and  $a_1$  so that*

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

*where  $y_1(x)$  is the solution satisfying the initial conditions  $y_1(x_0) = 1$ ,  $y_1'(x_0) = 0$  and  $y_2(x)$  is the solution satisfying the initial conditions  $y_2(x_0) = 0$ ,  $y_2'(x_0) = 1$ .*

EXAMPLE 8. *Given differential equation*

$$xy'' + y' + xy = 0.$$

- a) Seek power series solutions of this equation about  $x_0 = 1$ : find the recurrence relation for coefficients of the power series about  $x_0 = 1$  representing a solution (in general, a recurrence relation is a relation expressing the  $n$ th coefficients  $a_n$  in terms of some previous ones).

- b) Find the first five terms in the power expansion about  $x_0 = 1$  of the solution of the equation (1) satisfying initial conditions  $y(1) = 3$ ,  $y'(1) = 1$ .

**Radius of convergence as the minimal distance to singularities.**

There is a more elegant way to find the radius of convergence of the Taylor series at  $x_0$  of the analytic function  $f$  rather than calculating the coefficients of this Taylor series (i.e. derivatives of any order of  $f$  at  $x_0$ ). For this we pass to the complex plane.

Assume for simplicity that  $f(x) = \frac{Q}{P}$ , where  $Q$  and  $P$  are polynomials and all common linear factors (in general with complex coefficients) of  $Q$  and  $P$  are canceled. Then  $f$  is analytic at  $x_0$  if and only if  $P(x_0) \neq 0$  and **the radius of convergence of the Taylor series about  $x_0$  is equal to the distance to the nearest complex zero of  $P$ .**

EXAMPLE 9. Given  $f(x) = \frac{1}{1+x^2}$  what is the radius of convergence of the Taylor series of  $f$

a) around  $x_0 = 0$

b) around  $x_0 = 1$

c) around  $x_0 = 2$

REMARK 10. Note that the function  $f(x) = \frac{1}{1+x^2}$  of the previous example is defined for all real  $x$ , so we see the “problem” (non-convergence of Taylor series for  $|x| > 1$ ) only when passing to the complex plane. This is another example when the use of complex numbers is very natural and helpful.

EXAMPLE 11. Determine a lower bound for the radius of convergence of series solutions about each given point of the following equation:

a)  $xy'' + y' + xy = 0$  about  $x_0 = 1$  (as in Example 8)

b)  $(x^2 + 2x + 2)y'' + xy' + 4y = 0$  about

i)  $x_0 = 0$

ii)  $x_0 = -1$

iii)  $x_0 = -3$