27: The Phase Plane: Linear Systems (section 9.1)

1. Consider a homogeneous linear system of 2 equations

$$X' = AX, \quad X \in \mathbb{R}^2, \quad \det A \neq 0 \tag{1}$$

As in chapter 1 the points for which the right-hand side of the system is equal to zero are called equilibrium points. We also use the terminology of chapter 9 of the book where they are called **critical points** (another very common terminology is **stationary points**). For system (1) the condition $\det A \neq 0$ implies that $X = 0$ is the only its stationary point.

We want to know how the phase portrait (i.e the collection of integral curves/trajectories of system (1)) looks like for different A (of course it will depend on the type of eigenvalues and theit algebraic/geometric multiplicities).

Case 1. Distinct real eigenvalues.

2. Assume that λ_1 and λ_2 are the eigenvalues of matrix A so that they real and distinct. Assume that v^1 and v^2 are the corresponding eigenvectors. The general solution is

$$X(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2. \tag{2}$$

In order to understand how the curves given in parametric form (2) looks like in the (x_1, x_2)-plane let us first study how it looks like in "distorted" coordinates w.r.t. to the basis $\{v^1, v^2)$

<u>Explanation what does it mean coordinate with respect to a basis:</u>

3. In this "distorted" coordinate system the parametric form for the curve (2) is given by :

$$y_1(t) = C_1 e^{\lambda_1 t} \tag{3}$$

$$y_2(t) = C_2 e^{\lambda_2 t} \tag{4}$$

Assuming that $C_1 \neq 0$ express y_2 as a function of y_1:

So,

$$\boxed{y_2 = C y_1^{\alpha}}, \text{where } \alpha = \frac{\lambda_2}{\lambda_1}, \quad C = \frac{C_2}{C_1^{\lambda_2/\lambda_1}}, \text{ if } C_1 \neq 0, \qquad y_1 = 0, \text{ if } C_1 = 0. \tag{5}$$

Case 1a: Real Distinct Eigenvalues of the Same Sign $\Leftrightarrow \alpha > 0$-The Node

4. Then the phase portrait in (y_1, y_2)-plane looks like (taking also into account the arrows /the directions of motion along the trajectories):

5. in the original (x_1, x_2)- plane the phase portrait looks like:

6.

REMARK 1. *The trajectories that are (pieces of) straight lines lie on the eigenlines.*

REMARK 2. *The tangent lines to the trajectories near the origin are close to the eigenline corresponding to the eigenvalue which has smaller absolute value.*

Case 1b: Real Distinct Eigenvalues of Opposite signs $\Leftrightarrow \alpha < 0$-The Saddle

7. Then the phase portrait in (y_1, y_2)-plane looks like (taking also into account the arrows /the directions of motion along the trajectories):

Case 2: Complex Eigenvalues

8. Assume that $\lambda = \alpha + i\beta$ is a complex eigenvalue of A and for definiteness assume that $\beta > 0$. Assume that $v = \mathbf{a} + i\mathbf{b}$ is a corresponding eigenvector. Then the general solution is

$$\begin{aligned} &X(t) = C_1 \mathrm{Re}(e^{\lambda t} v) + C_2 \mathrm{Im}(e^{\lambda t} v) = \\ &e^{\alpha t}\big(C_1 \cos(\beta t) + C_2 \sin(\beta t)\big)\mathbf{a} + e^{\alpha t}\big(-C_1 \sin(\beta t) + C_2 \cos(\beta t)\big)\mathbf{b} \end{aligned} \tag{6}$$

9. So, in the coordinate (y_1, y_2) with respect to the basis $(\mathbf{a}, \mathbf{b})$ the trajectories of systems (1) are given by the following parametric equations:

$$y_1(t) = e^{\alpha t}\big(C_1 \cos(\beta t) + C_2 \sin(\beta t)\big) \tag{7}$$
$$y_2(t) = e^{\alpha t}\big(-C_1 \sin(\beta t) + C_2 \cos(\beta t)\big) \tag{8}$$

Using the same method as in section 15 (on complex eigenvalues) we can transform these equations to

$$y_1(t) = e^{\alpha t} R \cos(\beta t - \delta) \tag{9}$$
$$y_2(t) = -e^{\alpha t} R \sin(\beta t - \delta) \tag{10}$$

Subcase 2a: $\alpha = 0$-Center

10. Equations (9)-(10) have the form:

$$y_1(t) = R\cos(\beta t - \delta) \tag{11}$$

$$y_2(t) = -R\sin(\beta t - \delta) \tag{12}$$

This are the parametric equations of the circle in (y_1, y_2)-plane with the direction of motion ___________ and angular velocity β.

11. Returning to the original coordinates (x_1, x_2) circles are transformed to **ellipses** with *direction of motion from* **b** *to* **a** *in the shortest way*, because **b** corresponds to the positive direction of y_2-axis and **a** corresponds to the positive direction of y_1-axis in the (y_1, y_2)-plane.

REMARK 3. *The procedure how to determine the shape of these ellipses (i.e. the direction of the principle axis and the ratio of the semiaxis) is discussed in Enrichment 9.*

Subcase 2b: $\alpha < 0$-Spiral sink (or stable focus)

12. The amplitude decays exponentially $\Rightarrow$ Circles of the previous case are transformed to the spirals entering the origin with the same rule for the direction of motion as in the previous subcase.

Subcase 2c: $\alpha > 0$-Spiral source (or unstable focus)

13. The amplitude grows exponentially $\Rightarrow$ Circles of the previous case are transformed to the spirals going out of the origin with the same rule for the direction of motion as in the subcase 2a.

14. **Simple way to determine the direction of motion on the ellipses/ spirals (clockwise or counterclockwise) without calculation of an eigenvector**

Assume that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

has complex eigenvalues.

If $a_{21} < 0$,then the motion is clockwise and if $a_{21} > 0$ the motion is counterclockwise.

Similarly, *if $a_{12} > 0$,then the motion is clockwise and if $a_{12} < 0$ the motion is counterclockwise.*

EXPLANATION:

Case 3: Repeated Eigenvalues

Case 3a: Geometric multiplicity is equal to 2-Proper node/star point

15. If λ is eigenvalue of geometric multiplicity 2 , then the vectors of the standard basis e_1, e_2 (where $e_1 = (1,0)^T$ and $e_2 = (0,1)^T$) are eigenvectors and the general solution is

$$X(t) = \begin{pmatrix} C_1 e^{\lambda t} \\ C_2 e^{\lambda t} \end{pmatrix}$$

or , equialently $x_1(t) = C_1 e^{\lambda t}$, $x_2(t) = C_2 e^{\lambda t}$. Eliminating the parameter t, one get

$$x_2 = \frac{C_2}{C_1} x_1.$$

So, all trajectories lie on the straight lines through the origin

Case 3b: Geometric multiplicity is equal to 1-Improper node

16. Let v be a corresponding eigenvector, and w is a generalized eigenvector such that $v = (A - \lambda I)w$. Then the general solution is

$$X(t) = C_1 e^{\lambda t} v + C_2 e^{\lambda t}(w + tv) = e^{\lambda t}(C_1 + C_2 t)v + C_2 e^{\lambda t} w$$

So in the coordinates (y_1, y_2) with respect to the basis (v, w) the trajectories of the system (1) are given by the following parametric equations:

$$y_1(t) = C_1 e^{\lambda t}(C_1 + C_2 t) \tag{13}$$

$$y_2(t) = C_2 e^{\lambda_t} \tag{14}$$

One can express y_1 as a function of y_2 if $C_2 \neq 0$ (for details see handwritten notes):

$$y_1 = y_2(\alpha + \beta \ln |y_2|), \tag{15}$$

where $\beta = \frac{1}{\lambda}$ and $\alpha = \frac{C_1}{C_2} - \frac{1}{\lambda} \ln |C_2|$.

17. Then the trajectories will look as follows on (y_1, y_2) will look as follows:

18. Returning to the original (x_1, x_2)-plane v corresponds to the positive direction of y_1-axis and w corresponds to the positive direction of y_2-axis.

19.

REMARK 4. *The trajectories that are (pieces of) straight lines lie on the eigenline (generated by v).*

REMARK 5. *The tangent lines to the trajectories near the origin are close to the eigenline (generated by v.*

20. Simple rule to determine the shape of the trajectory: Assume that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

has a repeated eigenvalue with geometric multiplicity 1.

(a) *If $a_{21} < 0$ (or $a_{12} > 0$), then the direciton of motion on a part of the trajectory which is far from the origin is clockwise.*

(b) *If $a_{21} > 0$ (or $a_{12} < 0$), then the direction of motion on a part of the trajectory which is far from the origin is counterclockwise.*

EXPLANATION:

Summary of types and stability properties of critical poinits of planar homogeneous system with constant coefficients

21. Stability properties of linear systems $X' = AX$ with $\det(A - \lambda I) = 0$ and $\det A \neq 0$.

Eigenvalues, λ	Type of Critical Point	Stability
$\lambda_1 > \lambda_2 > 0$	Node(source)	Unstable
$\lambda_1 < \lambda_2 < 0$	Node (sink)	Asymptotically stable
$\lambda_2 < 0 < \lambda_1$	Saddle point	Unstable
$\lambda_{1,2} = \alpha \pm i\beta$, $\alpha > 0$	Spiral source	Unstable
$\lambda_{1,2} = \alpha \pm i\beta$, $\alpha < 0$	Spiral sink	Asymptotically stable
$\lambda_{1,2} = \alpha \pm i\beta$, $\alpha = 0$	Center	Stable
$\lambda_1 = \lambda_2 > 0$	Proper or Improper node(source)	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or Improper node(sink)	Asymptotically stable

Note that in the last two items of the table *proper node corresponds to the case when the geometric multiplicity is equal to* 2 *and improper one corresponds to the case when the geometric multiplicity is equal to* 1.

The case of nonhomogeneous linear system with constant coefficients

22. Consider the system

$$X' = AX + b, X, \quad b \in \mathbb{R}^2 \tag{16}$$

If again det $A \neq 0$, there is the unique critical point X^0, because the system $AX^0 + b = 0$ is equivalent to $AX^0 = -b$ and it has the unique solution. So, $b = -AX^0$ and (16) is equivalent to

$$X' = AX - AX^0 \Leftrightarrow (X - X^0)' = A(X - X^0) \tag{17}$$

Substitute $u = X - X^0$ to obtain the homogeneous linear system

$$U' = AU \tag{18}$$

23. $U(t)$ is a solution of (18) if and only if $x(t) = x^0 + u(t)$ is a solution of (16). Hence, the phase portrait of (16) is obtained from the phase portrait of (18) by *the shift (the translation) in* X^0:

Critical damping as a bifurcation (a qualitative change in the phase portrait under the small change of parameter

24. Damped unforced vibrations are given by the second order linear homogeneous equatiom

$$mu'' + \gamma u' + ku = 0 \tag{19}$$

The corresponding system is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k}{m}x_1 - \frac{\gamma}{m}x_2 \end{aligned} \tag{20}$$

Let γ_{crit} be the critical damping for (19), $\gamma_{crit} = 2\sqrt{mk}$ (saee section 16).

Then for the system (20)

(a) If $0 < \gamma < \gamma_{\text{crit}}$, then the origin is a spiral sink;

(b) If $\gamma = \gamma_{\text{crit}}$, then the origin is an improper nodal sink;

(c) If $\gamma > \gamma_{\text{crit}}$, then the origin is a nodal sink,

The phase portrait in each case are as follows and you can see how the nodal sink is transformed to the spiral sink through the improper nodal sink: