

27: The Phase Plane: Linear Systems (section 9.1)

1. Consider a homogeneous linear system of 2 equations

$$X' = AX, \quad X \in \mathbb{R}^2, \quad \det A \neq 0 \quad (1)$$

As in chapter 1 the points for which the right-hand side of the system is equal to zero are called equilibrium points. We also use the terminology of chapter 9 of the book where they are called **critical points** (another very common terminology is **stationary points**). For system (1) the condition $\det A \neq 0$ implies that $X = 0$ is the only its stationary point.

We want to know how the phase portrait (i.e the collection of integral curves/trajectories of system (1)) looks like for different A (of course it will depend on the type of eigenvalues and their algebraic/geometric multiplicities).

Case 1. Distinct real eigenvalues.

2. Assume that λ_1 and λ_2 are the eigenvalues of matrix A so that they are real and distinct. Assume that v^1 and v^2 are the corresponding eigenvectors. The general solution is

$$X(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2. \quad (2)$$

In order to understand how the curves given in parametric form (2) look like in the (x_1, x_2) -plane let us first study how it looks like in “distorted” coordinates w.r.t. to the basis $\{v^1, v^2\}$

Explanation what does it mean coordinate with respect to a basis:

3. In this “distorted” coordinate system the parametric form for the curve (2) is given by :

$$y_1(t) = C_1 e^{\lambda_1 t} \quad (3)$$

$$y_2(t) = C_2 e^{\lambda_2 t} \quad (4)$$

Assuming that $C_1 \neq 0$ express y_2 as a function of y_1 :

So,

$$\boxed{y_2 = C y_1^\alpha}, \text{ where } \alpha = \frac{\lambda_2}{\lambda_1}, \quad C = \frac{C_2}{C_1^{\lambda_2/\lambda_1}}, \text{ if } C_1 \neq 0, \quad (5)$$

$$y_1 = 0, \text{ if } C_1 = 0.$$

Case 1a: Real Distinct Eigenvalues of the Same Sign $\Leftrightarrow \alpha > 0$ -The Node

4. Then the phase portrait in (y_1, y_2) -plane looks like (taking also into account the arrows /the directions of motion along the trajectories):

5. in the original (x_1, x_2) - plane the phase portrait looks like:

6.

REMARK 1. *The trajectories that are (pieces of) straight lines lie on the eigenlines.*

REMARK 2. *The tangent lines to the trajectories near the origin are close to the eigenline corresponding to the eigenvalue which has smaller absolute value.*

Case 1b: Real Distinct Eigenvalues of Opposite signs $\Leftrightarrow \alpha < 0$ -The Saddle

7. Then the phase portrait in (y_1, y_2) -plane looks like (taking also into account the arrows /the directions of motion along the trajectories):

Case 2: Complex Eigenvalues

8. Assume that $\lambda = \alpha + i\beta$ is a complex eigenvalue of A and for definiteness assume that $\beta > 0$. Assume that $v = \mathbf{a} + i\mathbf{b}$ is a corresponding eigenvector. Then the general solution is

$$\begin{aligned} X(t) &= C_1 \operatorname{Re}(e^{\lambda t} v) + C_2 \operatorname{Im}(e^{\lambda t} v) = \\ &e^{\alpha t} (C_1 \cos(\beta t) + C_2 \sin(\beta t)) \mathbf{a} + e^{\alpha t} (-C_1 \sin(\beta t) + C_2 \cos(\beta t)) \mathbf{b} \end{aligned} \quad (6)$$

9. So, in the coordinate (y_1, y_2) with respect to the basis (\mathbf{a}, \mathbf{b}) the trajectories of systems (1) are given by the following parametric equations:

$$y_1(t) = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t)) \quad (7)$$

$$y_2(t) = e^{\alpha t}(-C_1 \sin(\beta t) + C_2 \cos(\beta t)) \quad (8)$$

Using the same method as in section 15 (on complex eigenvalues) we can transform these equations to

$$y_1(t) = e^{\alpha t} R \cos(\beta t - \delta) \quad (9)$$

$$y_2(t) = -e^{\alpha t} R \sin(\beta t - \delta) \quad (10)$$

Subcase 2a: $\alpha = 0$ -Center

10. Equations (9)-(10) have the form:

$$y_1(t) = R \cos(\beta t - \delta) \quad (11)$$

$$y_2(t) = -R \sin(\beta t - \delta) \quad (12)$$

This are the parametric equations of the circle in (y_1, y_2) -plane with the direction of motion _____ and angular velocity β .

11. Returning to the original coordinates (x_1, x_2) circles are transformed to **ellipses** with *direction of motion from \mathbf{b} to \mathbf{a} in the shortest way*, because \mathbf{b} corresponds to the positive direction of y_2 -axis and \mathbf{a} corresponds to the positive direction of y_1 -axis in the (y_1, y_2) -plane.

REMARK 3. *The procedure how to determine the shape of these ellipses (i.e. the direction of the principle axis and the ratio of the semiaxis) is discussed in Enrichment 9.*

Subcase 2b: $\alpha < 0$ -Spiral sink (or stable focus)

12. The amplitude decays exponentially \Rightarrow Circles of the previous case are transformed to the spirals entering the origin with the same rule for the direction of motion as in the previous subcase.

Subcase 2c: $\alpha > 0$ -Spiral source (or unstable focus)

13. The amplitude grows exponentially \Rightarrow Circles of the previous case are transformed to the spirals going out of the origin with the same rule for the direction of motion as in the subcase 2a.

14. **Simple way to determine the direction of motion on the ellipses/ spirals (clockwise or counterclockwise) without calculation of an eigenvector**

Assume that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

has complex eigenvalues.

If $a_{21} < 0$, then the motion is clockwise and if $a_{21} > 0$ the motion is counterclockwise.

Similarly, if $a_{12} > 0$, then the motion is clockwise and if $a_{12} < 0$ the motion is counterclockwise.

EXPLANATION:

Case 3: Repeated Eigenvalues

Case 3a: Geometric multiplicity is equal to 2-Proper node/star point

15. If λ is eigenvalue of geometric multiplicity 2, then the vectors of the standard basis e_1, e_2 (where $e_1 = (1, 0)^T$ and $e_2 = (0, 1)^T$) are eigenvectors and the general solution is

$$X(t) = \begin{pmatrix} C_1 e^{\lambda t} \\ C_2 e^{\lambda t} \end{pmatrix}$$

or, equivalently $x_1(t) = C_1 e^{\lambda t}$, $x_2(t) = C_2 e^{\lambda t}$. Eliminating the parameter t , one get

$$x_2 = \frac{C_2}{C_1} x_1.$$

So, all trajectories lie on the straight lines through the origin

Case 3b: Geometric multiplicity is equal to 1-Improper node

16. Let v be a corresponding eigenvector, and w is a generalized eigenvector such that $v = (A - \lambda I)w$. Then the general solution is

$$X(t) = C_1 e^{\lambda t} v + C_2 e^{\lambda t} (w + tv) = e^{\lambda t} (C_1 + C_2 t) v + C_2 e^{\lambda t} w$$

So in the coordinates (y_1, y_2) with respect to the basis (v, w) the trajectories of the system (1) are given by the following parametric equations:

$$y_1(t) = C_1 e^{\lambda t} (C_1 + C_2 t) \tag{13}$$

$$y_2(t) = C_2 e^{\lambda t} \tag{14}$$

One can express y_1 as a function of y_2 if $C_2 \neq 0$ (for details see handwritten notes):

$$y_1 = y_2 (\alpha + \beta \ln |y_2|), \tag{15}$$

where $\beta = \frac{1}{\lambda}$ and $\alpha = \frac{C_1}{C_2} - \frac{1}{\lambda} \ln |C_2|$.

17. Then the trajectories will look as follows on (y_1, y_2) will look as follows:

18. Returning to the original (x_1, x_2) -plane v corresponds to the positive direction of y_1 -axis and w corresponds to the positive direction of y_2 -axis.

19.

REMARK 4. *The trajectories that are (pieces of) straight lines lie on the eigenline (generated by v).*

REMARK 5. *The tangent lines to the trajectories near the origin are close to the eigenline (generated by v).*

20. Simple rule to determine the shape of the trajectory: Assume that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

has a repeated eigenvalue with geometric multiplicity 1.

- (a) *If $a_{21} < 0$ (or $a_{12} > 0$), then the direction of motion on a part of the trajectory which is far from the origin is clockwise.*
- (b) *If $a_{21} > 0$ (or $a_{12} < 0$), then the direction of motion on a part of the trajectory which is far from the origin is counterclockwise.*

EXPLANATION:

Summary of types and stability properties of critical points of planar homogeneous system with constant coefficients

21. Stability properties of linear systems $X' = AX$ with $\det(A - \lambda I) = 0$ and $\det A \neq 0$.

Eigenvalues, λ	Type of Critical Point	Stability
$\lambda_1 > \lambda_2 > 0$	Node(source)	Unstable
$\lambda_1 < \lambda_2 < 0$	Node (sink)	Asymptotically stable
$\lambda_2 < 0 < \lambda_1$	Saddle point	Unstable
$\lambda_{1,2} = \alpha \pm i\beta, \alpha > 0$	Spiral source	Unstable
$\lambda_{1,2} = \alpha \pm i\beta, \alpha < 0$	Spiral sink	Asymptotically stable
$\lambda_{1,2} = \alpha \pm i\beta, \alpha = 0$	Center	Stable
$\lambda_1 = \lambda_2 > 0$	Proper or Improper node(source)	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or Improper node(sink)	Asymptotically stable

Note that in the last two items of the table *proper node* corresponds to the case when the geometric multiplicity is equal to 2 and *improper one* corresponds to the case when the geometric multiplicity is equal to 1.

The case of nonhomogeneous linear system with constant coefficients

22. Consider the system

$$X' = AX + b, X, \quad b \in \mathbb{R}^2 \quad (16)$$

If again $\det A \neq 0$, there is the unique critical point X^0 , because the system $AX^0 + b = 0$ is equivalent to $AX^0 = -b$ and it has the unique solution. So, $b = -AX^0$ and (16) is equivalent to

$$X' = AX - AX^0 \Leftrightarrow (X - X^0)' = A(X - X^0) \quad (17)$$

Substitute $u = X - X^0$ to obtain the homogeneous linear system

$$U' = AU \quad (18)$$

23. $U(t)$ is a solution of (18) if and only if $x(t) = x^0 + u(t)$ is a solution of (16). Hence, the phase portrait of (16) is obtained from the phase portrait of (18) by *the shift (the translation) in X^0* :

Critical damping as a bifurcation (a qualitative change in the phase portrait under the small change of parameter)

24. Damped unforced vibrations are given by the second order linear homogeneous equation

$$mu'' + \gamma u' + ku = 0 \quad (19)$$

The corresponding system is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k}{m}x_1 - \frac{\gamma}{m}x_2 \end{aligned} \quad (20)$$

Let γ_{crit} be the critical damping for (19), $\gamma_{\text{crit}} = 2\sqrt{mk}$ (see section 16).

Then for the system (20)

- (a) If $0 < \gamma < \gamma_{\text{crit}}$, then the origin is a spiral sink;
- (b) If $\gamma = \gamma_{\text{crit}}$, then the origin is an improper nodal sink;
- (c) If $\gamma > \gamma_{\text{crit}}$, then the origin is a nodal sink,

The phase portrait in each case are as follows and you can see how the nodal sink is transformed to the spiral sink through the improper nodal sink: