28: Nonlinear systems: stability (section 9.2) and phase portrait (section 9.3)

1. Consider an arbitrary system of n first order differential equations:

$$X' = F(X),\ X \in \mathbb{R}^n \quad \Leftrightarrow \quad \begin{cases} x_1' = F_1(x_1, \dots, x_n), \\ x_2' = F_2(x_1, \dots, x_n), \\ \quad \vdots \\ x_n' = F_n(x_1, \dots, x_n) \end{cases} \tag{1}$$

DEFINITION 1. *A point* $X^0 = (x_1^0, \dots, x_n^0) \in \mathbb{R}^n$ *is called a* critical (also stationary or equilibrium) point of system (1), *if*

$$F(X^0) = 0 \quad \Leftrightarrow \quad \begin{cases} F_1(x_1^0, \dots, x_n^0) = 0, \\ F_2(x_1^0, \dots, x_n^0) = 0, \\ \quad \vdots \\ F_n(x_1^0, \dots, x_n^0) = 0 \end{cases} \tag{2}$$

2.

DEFINITION 2. *A critical point* X^0 *is called* stable *if for any neighbourhood* [1] *of* X^0 *there exists another (maybe smaller) neighbourhood* V *of* X^0 *such that if trajectory* $X(t)$ *of* (1) *starts in* V *(i.e.* $X(0) \in V$*), then it stays in* U *for any* $t \geq 0$.

3.

DEFINITION 3. *A critical point* X^0 *is called* unstable, *if it is not stable or equivalently, making the negation of Definition 2: if there exist a neighbourhood* U *of* X^0 *such that for any neighbourhood* V *of* x^0 *there exists a trajectory* $X(t)$ *starting at* V *but going out of* U *for some* $t > 0$.

[1] By a neighbourhood of X^0 we mean a ball around X^0 (a disk if $n = 2$ and an interval if $n = 1$).

4.

DEFINITION 4. *A critical point X^0 is called* asymptotically stable*, if it is stable and there exists a neighbourhood V of X^0 such that for any trajectory $X(t)$ starting at V, we have $X(t) \to X^0$ as $t \to +\infty$. In addition, it is called* globally asymptotically stable *if one can take $V = \mathbb{R}^n$, i.e. $X(t) \to X^0$ as $t \to +\infty$ for any trajectory.*

REMARK 5. *A center in linear planar system is an example of a stable critical point which is not asymptotically stable.*

Stability of the origin for Linear Systems

5. Consider a linear system

$$X' = AX, \quad X \in \mathbb{R}^n, \quad \det A \neq 0. \tag{3}$$

As was mentioned in the previous notes, in this case $X^0 = 0$ is the only critical point.

The set of all eigenvalues of a matrix A is called the *spectrum* of A and is denoted by $\operatorname{Spec}(A)$. From the form of solutions of (3) discussed in chapter 7 of the book we get:

THEOREM 6. *In thre case when* $\det A \neq 0$ *the following statements hold:*

(a) $X^0 = 0$ is (globally) asymptotically stable if and only if the real part of every eigenvalue of A is negative , i.e. the spectrum of A lies in the the left half-plane of the complex plane (but not on the imaginary axis);

(b) $X^0 = 0$ *is stable if and only if the real part of every eigenvalue of* A *is nonpositive, i.e. the spectrum of* A *lies in the the left half-plane of the complex plane or on the imaginary axis, and for each eigenvalue on the imaginary axis the geometric multiplicity is equal to the algebraic multiplicity.*

(c) $X^0 = 0$ *is unstable if and only if either there exists an eigenvalue of* A *with positive real part or there exists an eigenvalue on the imaginary axis with the geometric multiplicity being strictly less than the geometric multiplicity.*

6.

COROLLARY 7. *In the case of* $n = 2$, $\det A \neq 0$ *the conditions (b) and (c) of the previous Theorem are equivalent to the following:*

(a) $X^0 = 0$ *is stable if and only if the real part of every eigenvalue of* A *is nonpositive, i.e. the spectrum of* A *lies in the the left half-plane of the complex plane or on the imaginary axis.*

(b) $X^0 = 0$ *is unstable if and only if there exists an eigenvalue of* A *with positive real part.*

Nonlinear (locally linear) Systems: Linearization Principle

7. Let for simplicity $n = 2$. We will use coordinates (x, y) in $\mathbb{R}^2$. Then system (1) consists of two equations:

$$\begin{cases} x' = f(x, y), \\ y' = g(x, y), \end{cases} \tag{4}$$

Assume that (x_0, y_0) is a stationary point of (4), i.e.

$$\begin{cases} f(x_0, y_0) = 0, \\ g(x_0, y_0) = 0, \end{cases} \tag{5}$$

Expand functions f and g into the Taylor expansions around (x_0, y_0) up to a linear term:

8. Substituting these expansions into the right hand-side of (4) and ignoring the remaining terms (as terms of smaller order than linear terms near (x_0, y_0)) we get

9. Making a substitution $u = x - x_0$, $v = y - y_0$ we get the following system:

$$\begin{cases} u' = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v, \\ v' = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v, \end{cases} \tag{6}$$

This system is a linear system in u and v with the matrix

$$J(x_0, y_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}. \tag{7}$$

DEFINITION 8. *System* (6) *is called the* linearization *of system* (4) *at the critical point* (x_0, y_0). *The matrix* (7) *is called the* Jacobi matrix *of system* (4) *at the point* (x_0, y_0).

10. The intuition is that since we ignore very small terms when passing from (4) to (6) the behaviour of the trajectories of the original nonlinear system (4) near (x_0, y_0) is similar to the behaviour of the trajectories of its linearization (6) and the latter we studied in the previous section based on Chapter 7. However, we need an extra assumption on the eigenvalues of the Jacobi matrix $J(x_0, y_0)$.

    THEOREM 9 (**Linearization Principle**). *If the Jacobi matrix* $J(x_0, y_0)$ *of a critical point* (x_0, y_0) *of system* (4) *does not have eigenvalues on the imaginary axis (i.e. no eigenvalues with zero real part), then the stability properties of* (x_0, y_0) *are the same as stability property of the origin of the origin for its linearization* (6) *at* (x_0, y_0).

    REMARK 10. *Note that we already proved the Linearization Principle in the case* $n = 1$, *when we discussed the phase line (see section 8, item 6 there).*

11. Under the same assumptions as in Theorem 9 the phase portrait of the nonlinear system (1) near the criticical point (x_0, y_0) looks similar (as a small distortion) of the phase portrait of the linearization (6).

    For example, if the Jacobi matrix $J(x_0, y_0)$ has two real eigenvalues λ_1 and λ_2 of opposite signs, $\lambda_1 < 0$ and $\lambda_2 > 0$, then there exist two curves Γ_- amd Γ_+ in a neighbourhood U of (x_0, y_0) passing through (x_0, y_0) and satisfying the following properties:

    (a) If $\big(x(0, y(0)\big) \in \Gamma_+$, then $\big(x(t), y(t)\big) \to (x_0, y_0)$ and $t \to +\infty$;

    (b) If $\big(x(0, y(0)\big) \in \Gamma_-$, then $\big(x(t), y(t)\big) \to (x_0, y_0)$ and $t \to -\infty$.

    In addition, the tangent line to the curve Γ_- at (x_0, y_0) coincides with the eigenline of $\lambda_1 < 0$ and the tangent line to the curve Γ_+ at (x_0, y_0) coincides with the eigenline of $\lambda_2 > 0$.

    The curves Γ_- and Γ_+ are called the *stable* and *unstable separatrices* of the saddle critical point (x_0, y_0), respectively. The phase portrait of the original system (4) near (x_0, y_0) is a "distortion" of the plane portrait of its linearization , i.e. looks as the phase portrait of the saddle:

12. Why the case of $\text{Re}\lambda = 0$ is excluded from Theorem (9)? This cases is sensitive to nonlinear perturbations as a threshold between stability and instability, so in this case the stability cannot be decided by the linearization as the following example shows:

    EXAMPLE 11. *Compare stability properties of the origin of the system*

$$\begin{cases} x' = y, \\ y' = -x + \varepsilon y^3, \end{cases} \tag{8}$$

    and its linearization for all values of parameter ε.

13.

EXAMPLE 12 (a model of competing species, section 9.4). *Consider the following system*

$$\begin{cases} x' &= x(7 - x - 2y) \\ y' &= y(5 - y - x) \end{cases} \tag{9}$$

(a) Why is this system related to competing species?

(b) Determine all critical points of (9).

(c) For each critical point find the corresponding linearization and determine the type of each critical point and their stability properties (i.e. whether they are stable, asymptotically stable, or unstable)?

(d) Sketch the phase portrait of system (9) in the first quadrant.

(e) System (9) corresponds to a model of competing species. Based on your analysis in the previous item, answer the following question: does the coexistence occurs in the model given by system (9)?

Discussion of the general competing species model (section 9.4)

14. Consider

$$\begin{cases} x' &= x(\varepsilon_1 - \sigma_1 x - \alpha_1 y) \\ y' &= y(\varepsilon_2 - \sigma_2 y - \alpha_2 x) \end{cases} \tag{10}$$

under assumption that the algebraic linear system

$$\begin{cases} \sigma_1 x + \alpha_1 y = \varepsilon_1 \\ \alpha_2 x + \sigma_2 y = \varepsilon_2 \end{cases} \tag{11}$$

has exactly one solution and it lies in the first quadrant. In particular, this implies $\sigma_1\sigma_2 - \alpha_1\alpha_2 \neq 0$.

15. the following two lines play important role:

    (a) *x-nullcline* is given by the equation $\sigma_1 x + \alpha_1 y = \varepsilon_1$;

    (b) *y-nullcline* is given by the equation $\alpha_2 x + \sigma_2 y = \varepsilon_2$;

    Consider the following two cases separately:

16. **Case 1**: $\sigma_1\sigma_2 - \alpha_1\alpha_2 < 0$ (no co-existence)

17. **Case 1**: $\sigma_1\sigma_2 - \alpha_1\alpha_2 > 0$ (co-existence)