8. Qualitative analysis of autonomous equations on the line/population dynamics models, phase line, and stability of equilibrium points (corresponds to section 2.5)

1. Equation $y' = f(t, y)$ is called **autonomous** if f does not depend on time t. In other words, the equation can be written as

$$y' = f(y). \tag{1}$$

   Note that this equation is separable but out task is not to solve it explicitly but to analyze to what limits the solutions approach as t tens to $+\infty$ or to $-\infty$.

2. We assume that f and $\dfrac{df}{dy}$ are continuous in order to guarantee the existence and uniqueness of solutions of initial value problems.

   In the sequel we will strongly use the following geometric interpretation of the uniqueness of an initial value problem: *two different integral curves (i.e. the graphs of the solutions in (t, y)-plane) never intersect each other.*

3. Given the differential equation:

$$y' = y - y^2 \tag{2}$$

   (a) Find all equilibrium points.

   (b) Sketch a direction field.

(c) Based on the sketch of the direction field from the item (b) answer the following questions:

i. Let $y(t)$ be the solution of equation (2) satisfying the initial condition $y(0) = \frac{1}{2}$. Find the limit of $y(t)$ when $t \to +\infty$ and the limit of $y(t)$ when $t \to -\infty$ (for this you do not need to find $y(t)$ explicitly).

ii. Find all y_0 such that the solution of the equation (2) with the initial condition $y(0) = y_0$ has the same limit at $+\infty$ as the solution from the item (c)i.

iii. Let $y(t)$ be the solution of equation (2) with $y(0) = -1$. Decide wether $y(t)$ is monotonically decreasing or increasing and find to what value it approaches when t increases (the value might be infinite).

(d) Find the solution of the equation (2) with $y(0) = -1$ explicitly. Determine the interval in which this solution is defined.

4. **Phase line and phase portrait on it**: For autonomous equation instead of (t, y)-plane, everything can be seen on the y-axis, which is called the *phase line*. In general, the *phase portrait* is the sketch on the phase space of the trajectories y(t) of the solutions of the equation/system together with indication by arrows of the direction of motion (as t increases). For the phase line the trajectories occupy some segments of this line between equilibrium points or they are just equilibrium points, so only the arrows matter.

   If y_0 is an equilibrium point then the trajectory passing through y_0 coincides with this point (i.e $y(t) \equiv y_0$). Also on each interval between two consecutive equilibrium points the slope $f(y)$ has the same sign, i.e. the direction of motion of trajectories in this interval along the y-axis is fixed.

   So, to sketch the phase portrait on the phase line (assuming that the phase line is drawn horizontally with positive direction to the right) you need to

   (a) find all equilibrium points (i.e. roots of $f(y) = 0$) and mark those points on the phase line;

   (b) determine the sign of $f(y)$ on each interval between the consecutive equilibrium points and put the corresponding arrows indicating the direction of motion of the trajectories of the equation in this interval. In other words, over those intervals where $f(y) > 0$ draw arrows pointing to the right, and on those where $f(y) < 0$ draw arrows pointing to the left (indicating in which direction are solutions flowing).

5. Sketch the phase portrait for the equation (2).

6. **Classification of equilibrium points on the phase line regarding the question of stability** This sketch of the phase portrait is very helpful in determining the **stability properties** of the equilibrium points.

   Namely if one passes through an (isolated) equilibrium point y_0 in the positive direction along y-axis, the following three cases are possible

   1. $f(y)$ changes sign from "+" to "−". In this case all trajectories starting at points near y_0 will approach y_0 and the equilibrium point y_0 is said to be **asymptotically stable**.

      Note that if $f'(y_0) < 0$, then y_0 is asymptotically stable.
   2. $f(y)$ changes sign from "−" to "+". In this case all trajectories starting at points near y_0 will go away from y_0 and the equilibrium point y_0 is said to be **unstable**. Note that

      if $f'(y_0) > 0$, then y_0 is unstable.
   3. $f(y)$ does not change sign as y passes through y_0 . In this case all nearby trajectories from one side of y_0 approach y_0 and all nearby trajectories from another side of y_0 go away from y_0 and the equilibrium point y_0 is said to be **semistable** (the latter terminology is not standard and makes sense only on the phase line but not , for example, on the phase plane, i.e. for systems of two first order ODEs).

7. Determine the stability properties of the equilibrium points of the equation (2).

8. *Exponential GROWTH* Let $y = y(t)$ be the population of the given species at time t. Hypothesis: $\frac{\mathrm{d}y}{\mathrm{d}t} = ry$, where r is called the **rate of intrinsic growth** (i.e. no competition

between individuals is taken into account .

9. *LOGISTIC GROWTH:* the growth rate depends on the population (replace r by a function $h(y)$): $\frac{\mathrm{d}y}{\mathrm{d}t} = h(y)y$
   Verhulst's model: $h(y) = r - ay$ $(r, a > 0)$: i.e. the model is given by the equation
   $$y' = ry - ay^2.$$
   The term $-ay^2$ describes the competitions between individuals (it is proportional to the number of pairs of individuals).

   The stable equilibrium point $K = \frac{r}{a}$ is called **saturation level** or the **enviromental carrying capacity**.