9: Systems of FIRST Order Equations and their relation to higher order equations (section 7.1)

1. A first order system of ordinary differential equations (ODEs):

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \dots, x_n) \\ x_2' &= F_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \dots, x_n) \end{aligned} \tag{1}$$

2. A set of differentiable functions $x_1(t), x_2(t), \dots, x_n(t)$ satisfying the system (1) is called a **solution** of the system (1).

3. System of ODE using a **vector notation**:

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F_1(t, x_1, x_2, \dots, x_n) \\ F_2(t, x_1, x_2, \dots, x_n) \\ \vdots \\ F_n(t, x_1, x_2, \dots, x_n) \end{pmatrix} \tag{2}$$

Then the system (1) can be written as

$$\mathbf{X}' = \mathbf{F}(t, \mathbf{X}). \tag{3}$$

Symbolically this is exactly the same expression as for a single first order equation.

DEFINITION 1. *The system* (1) *is called* <u>*autonomous*</u> *if the right-hand side of it is independent of* t*, i.e. is of the form* $\mathbf{F}(\mathbf{X})$,

$$\mathbf{X}' = \mathbf{F}(x). \tag{4}$$

and non-autonomous otherwise.

REMARK 2. *Autonomous systems are special first order systems. However, any non-autonomous system on* n *unknown functions* $(x_1, \dots, x_n)$ *of* t *can be seen as an autonomous system in* $n+1$*-unknown functions* $(x_0, x_1, \dots, x_{n+1})$ *such that* $x_0'(t) = 1$*. Namely, we consider the following system of* $n+1$ *equations of* $n+1$ *unknown functions:*

The very first equation of this system implies that if $x_0(0) = 0$ then $x_0(t) = t$ and then the column vector function X as in (2) is the solution of the original system (1). Therefore conceptually we can restrict ourselves to autonomous systems only.

4. **Vector fields and autonomous first order systems:**

5. A *vector field* F on $\mathbb{R}^n$: at each point $\mathbf{X}$ of $\mathbb{R}^n$ a vector $\mathbf{F}(\mathbf{X})$ starting at this point $\mathbf{X}$ is given.

6. An *integral curve* (*integral trajectory* $\mathbf{X}(t)$ of a vector field $\mathbf{F}$ is a curve in $\mathbb{R}^n$ such that the velocity $\mathbf{X}'(t)$ to this curve at every its point $\mathbf{X}(t)$ (or, equivalently, at every time moment t) coincides with the vector fields $\mathbf{F}$ at this point, i.e. with the vector $\mathbf{F}\left(\mathbf{X}(t)\right)$.

   In other words,

$$\mathbf{X}'(t) = \mathbf{F}\left(\mathbf{X}(t)\right)$$

   i.e. $\mathbf{X}(t)$ is an integral curve of the field F if and only if it is a solution of the autonomous equation (4).

   REMARK 3. *One can define the analog if direction field for nonautonomous system* (1)*: it is a direction field in* $\mathbb{R}^{n+1}$ *such that the line segment is generated by the vector field of the corresponding autonomous system in* $\mathbb{R}^{n+1}$ *as described in Remark 2.*

7. To any autonomous system of n equations with n unknown function one can assign a vector field in $\mathbb{R}^n$ and vice versa, to any vector field $\mathbf{F}$ in $\mathbb{R}^n$ corresponds an autonomous system (4). Then $\mathbb{R}^n$ is called the *phase space* of the system (4). More generally, the vector field may be defined not on the whole $\mathbb{R}^n$ but in some region R of $\mathbb{R}^n$ or , for example, on a surface or higher dimensional analog of a surface S (like a sphere, a torus etc, depending on a model) in $\mathbb{R}^n$ (in the latter case the vector field must be tangent to such surface). In this case the sets R and S are also called phase spaces.

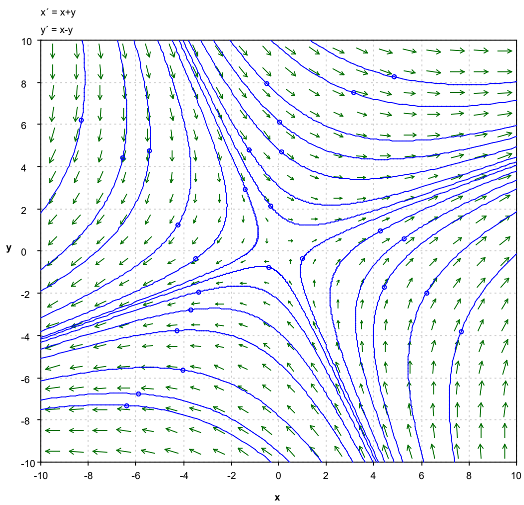
   The *phase portrait* of a system is a representative sketch of integral curves of the system on the phase space.

8. For $n = 1$ we have just one equation and the phase space in the autonomous case is just a line $\mathbb{R}$ (what we called the phase line in the previous section). The nonzero vectors tangent to $\mathbb{R}$ have only two directions, positive and negative, so those directions were the only what matters when we analyzed the behaviour of the solutions. For $n > 1$ there are much richer variety of how the phase portrait may look like compared to the case when $n = 1$.

   Here are several examples of sketch of the vector fields corresponding to a given system of two equations using **pplane**. The corresponding phase portraits can be sketched by drawing several representative phase lines. I mark each example with certain name but at this moment you do not have to put any attention on those names

   EXAMPLE 4 (saddle point).

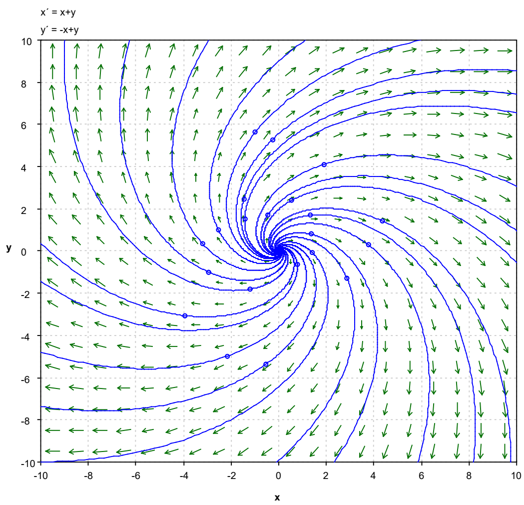
$$\begin{cases} x_1' & = & x_1 + x_2 \\ x_2' & = & x_1 - x_2 \end{cases}$$



   [spiral source]

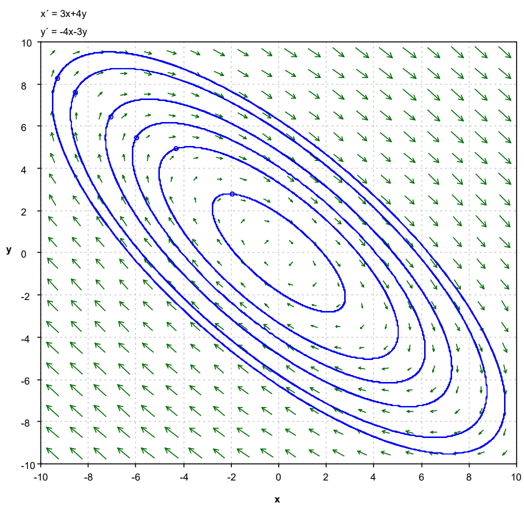
   EXAMPLE 5.

$$\begin{cases} x_1' & = & x_1 + x_2 \\ x_2' & = & -x_1 + x_2 \end{cases}$$



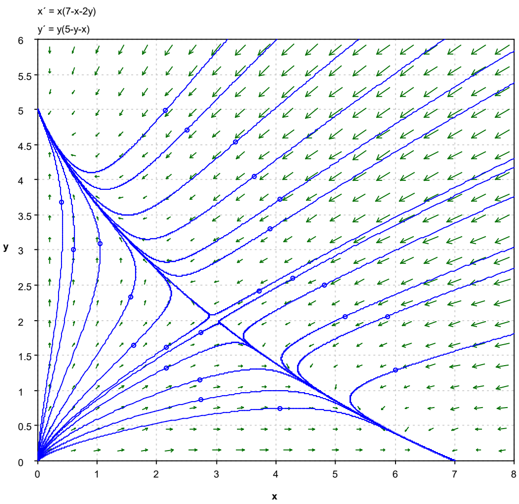
EXAMPLE 6 (center).

$$\begin{cases} x_1' &= 3x_1 + 4x_2 \\ x_2' &= -4x_1 - 3x_2 \end{cases}$$



EXAMPLE 7 (competing species).

$$\begin{cases} x' &= x(7-x-2y) \\ y' &= y(5-y-x) \end{cases}$$



We will learn how to solve explicitly systems in Examples 4-6, which are linear homogeneous systems with constant coefficients and how to analyze the nonlinear system in Example 7 based on the theory of linear systems (without the knowledge of this theory the software will not be really useful).

9. **Existence and Uniqueness Theorem** for IVP defined by a system: Consider the IVP:

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \dots, x_n) \\ x_2' &= F_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \dots, x_n) \\ x_1(t_0) &= x_1^0 \\ x_2(t_0) &= x_2^0 \\ &\vdots \\ x_n(t_0) &= x_n^0 \end{aligned} \tag{5}$$

is literally the same as Theorem 3 in section 7 of the notes devoted to single equation: *If each of the functions $F_1, F_2, \dots, F_n$ and the partial derivatives* $\frac{\partial F_1}{\partial x_k}, \frac{\partial F_2}{\partial x_k}, \dots, \frac{\partial F_n}{\partial x_k}$ $(1 \le k \le n)$

are continuous in a region

$$R = \{\alpha < t < \beta, \alpha_1 < x_1 < \beta_1, \alpha_2 < x_1 < \beta_2, \dots, \alpha_n < x_n < \beta_n\}$$

and the point $(t_0, x_1^0, \dots, x_n^0)$ *belongs to* R*, then there is an interval* $(t_0 - h, t_0 + h)$ *in which there exists a unique solution of the IVP (5).*

How to transform a scalar ODE or order n to a system of n first order equations

10. Any scalar ODE equation of order n,

$$y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)})$$

can be transformed to a system of n DE of the first order by introducing derivatives up to order $n-1$ as new variables.

11. To transform the following n-th order IVP,

$$\begin{aligned} &y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)}), \\ &y(t_0) = \alpha_0, \quad y'(t_0) = \alpha_1, \dots, \quad y^{(n-1)}(t_0) = \alpha_{n-1} \end{aligned} \tag{6}$$

into the system of first order equations we set

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \\ &\vdots \\ x_n(t) &= y^{(n-1)}(t) \end{aligned}$$

to get

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_n' &= f(t, x_1, x_2, \dots, x_n) \end{aligned} \tag{7}$$

subject to

$$x_1(t_0) = \alpha_0, \quad x_2(t_0) = \alpha_1, \dots, \quad x_n(t_0) = \alpha_{n-1}.$$

12. Consider the following ODE of unforced undamped vibration:

$$y'' + y = 0. \tag{8}$$

Transform (8) into a system of first order ODE. Is the obtained system autonomous?

13. Transform the equation

$$y^{(3)} + (\sin t)y'' + e^t((y')^2 + y^2) = 0$$

to the system of differential equations.

14. An obvious but very important remark is:

REMARK 8. *Afunction $y(t)$ is a solution of the equation* (6) *if an only if*

$$\mathbf{X}(t) = \begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix}$$

is a solution of the corresponding first order system (7).

This simple passage from single differential equation (6) of higher order to first-order system (7) allows us to consider the theory of second order and even higher order equations discussed in chapter 3 and 4 of the textbook as a particular case of the theory of systems of first order equation discussed in chapter 7. So, instead of first covering chapter 3 and then repeating the same in more general setting of chapter 7 we will start now with chapter 7 and treat the material of chapter 3 and (and even of chapter 4) simultaneously.