10. Linear Systems of ODEs, Matrix multiplication, superposition principle (parts of sections 7.2-7.4)

1. When each of the functions $F_1, F_2, \dots, F_n$ in right-hand side of the system of ODE (see equation (1)) of the previous section) is linear in the dependent variables $x_1, \dots, x_n$, we get a system of linear equations:

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + g_1(t) \\ x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + g_n(t) \end{aligned} \tag{1}$$

   When $g_k(t) \equiv 0$ $(1 \le k \le n)$, the linear system (1) is said to be **homogeneous**; otherwise it is **nonhomogeneous.**

2. In the Examples 4-6 of the previous set of notes the systems are linear homogeneous and also autonomous (which is equivalent in linear case to being with constant coefficients):

3. Consider the system

$$\begin{cases} x_1' &= x_2 \\ x_2' &= -x_1 \end{cases}$$

   Recall that this is the system of first order equations to which one transforms the equation of harmonic oscillator $y'' + y = 0$. What are the coefficients $p_{ij}(t)$ for this system ? Is it homogeneous?

4. **Existence and Uniqueness Theorem** for linear IVP is absolutely analogous to Theorem 4 of section 7 of the notes : *If all functions $p_{11}, p_{12}, \dots, p_{nn}$ and $g_1, \dots, g_n$ are continuous on an open interval $I = \{t : \alpha < t < \beta\}$, then there exists a unique solution of the system (1) that also satisfies the initial conditions $x_1(t_0) = x_1^0, x_2(t_0) = x_2^0, \dots, x_n(t_0) = x_n^0$, where t_0 is any point of I. Moreover, the solution exists throughout the interval I.*

Basic on Matrices and Matrix Multiplication

The language of matrices and operations between them is very natural in context of linear system of algebraic and differential equations. So this is an appropriate time to introduce some basics of matrix arithmetics.

5. An $m \times n$ (this is often called the size or dimension of the matrix) matrix is a table with m rows and n columns and the entry in the i-th row and j-th column is denoted by a_{ij}:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

6. A matrix is usually denoted by a capital letter and its elements by small letters : a_{ij} = entry in the ith row and jth column of A.

7. Two matrices are said to be **equal** if they have the same size and each corresponding entry is equal.

8. Special Matrices:

- A **square** matrix is any matrix whose size (or dimension) is $n \times n$ (i.e. it has the same number of rows as columns.) In a square matrix the diagonal that starts in the upper left and ends in the lower right is often called the **main diagonal.**
- The **zero** matrix is a matrix all of whose entries are zeroes.
- The **identity** matrix is a square $n \times n$ matrix, denoted I_n, whose main diagonal consists of all 1's and all the other elements are zero:

$$I_n = \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

- The **diagonal** matrix is a square $n \times n$ matrix of the following form

$$\begin{pmatrix} \lambda_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \lambda_n \end{pmatrix}$$

- **Column matrix** (=column vector) and the **row matrix** (=row vector) are those matrices that consist of a single column or a single row respectively:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & y_2 & \dots & y_n \end{pmatrix}$$

  Note that an n-dimensional column vector is an $n \times 1$ matrix, and an n-dimensional row vector is an $1 \times n$ matrix.

- **Transpose of a Matrix**: If A is an $m \times n$ matrix with entries a_{ij} , then A^T is the $n \times m$ matrix with entries a_{ji}.

  A^T is obtained by interchanging rows and columns of A.

9. **Matrix Arithmetic**

- The **sum** or difference of two matrices of the same size is a new matrix of identical size whose entries are the sum or difference of the corresponding entries from the original two matrices. Note that we cant add or subtract entries with different sizes.
- The **scalar multiplication** by a constant gives a new matrix whose entries have all been multiplied by that constant.
- If A, B, and C are matrices of the *same* size, then
  (a) $A + B = B + A$ (Commutative Law)
  (b) $(A + B) + C = A + (B + C)$ (Associative Law)
- **Matrix multiplication**: If Y is a row matrix of size $1 \times n$ and X is a column matrix

of size $n \times 1$ (see above), then the **matrix product** of Y and X is defined by

$$YX = \begin{pmatrix} y_1 & y_2 & y_3 & \cdots & y_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = y_1x_1 + y_2x_2 + y_3x_3 + \cdots + y_nx_n$$

- If A is an $m \times p$ matrix and matrix B is $p \times n$, then the product AB is an $m \times n$ matrix, and its element in the ith row and jth column is the product of the ith row of A and the jth column of B.
- RULE for multiplying matrices:

  The row of the 1st matrix must be the same size as the columns of the 2nd matrix or the number of columns of the 1st matrix should be equal to the number of rows of the 2nd matrix.

10. (a) Given

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ -1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -2 \\ 3 & 4 \\ 1 & 2 \end{pmatrix}$$

Compute $A - 2B$.

(b) Let $A = \begin{pmatrix} 1 & 2 & -3 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 0 & 1 & -1 \end{pmatrix}$. Find BA^T.

(c) *Example.* Given

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ -1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}$$

Compute AB and BA when it is possible.

(d) Compute AX if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ -1 & -2 & -3 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

11. FACTS and LAWS FOR MATRIX MULTIPLICATION: If the size requirements are met for matrices A, B and C, then
    - $AB \neq BA$ (NOT always Commutative)[1]
    - $A(B+C) = AB + AC$ (always Distributive)
    - $(AB)C = A(BC)$ (always Associative)
    - $AB = 0$ does not imply that $A = 0$ or $B = 0$.
    - $AB = AC$ does not imply that $B = C$.
    - $I_n A = A I_n = A$ for any square matrix A of size n.
12. A system of linear (algebraic) equations can be written as a matrix equation $AX = B$.

13. *Example.* Express the following system of linear (algebraic) equations in matrix form:

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & - & 7x_3 & = & 6 \\ -x_1 & - & 3x_2 & + & 11x_3 & = & 0 \\ & - & x_2 & + & x_3 & = & 1 \end{array}$$

[1] Since the multiplication of matrices is NOT commutative, you MUST multiply left to right.

Matrix Form of A System of linear differential equations

14. If X, $P(t)$, and $G(t)$ denote the respective matrices

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \dots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \dots & p_{2n}(t) \\ \vdots & & & \vdots \\ p_{n1}(t) & p_{n2}(t) & \dots & p_{nn}(t) \end{pmatrix}, \quad G(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix},$$

then the system of linear first-order ODE (1) can be written as

$$\mathbf{X}' = P\mathbf{X} + G.$$

If the system is homogeneous, its matrix form is then

$$\mathbf{X}' = P\mathbf{X}.$$

15. *Example.* Express the given system in matrix form:

(a) $$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 \end{aligned}$$

(b) $$\begin{aligned} x_1' &= x_2 - x_1 + t \\ x_2' &= -x_1 + 7x_2 - x_3 - e^t \\ x_3' &= 2x_2 - x_3 + \sin t \end{aligned}$$

Linear homogeneous systems of differential equations

16. Consider a system of first order linear homogeneous ODEs:

$$\mathbf{X}' = P(t)\mathbf{X}. \tag{2}$$

Superposition Principle: *If the vector functions $\mathbf{X}_1$ and $\mathbf{X}_2$ are solutions of the homogeneous system (2), then the function $\mathbf{X}(t) = C_1\mathbf{X}_1(t) + C_2\mathbf{X}_2(t)$ is also a solution for any constants C_1, C_2.*

An expression $C_1\mathbf{X}_1 + C_2\mathbf{X}_2$, where $\mathbf{X}_1$, $\mathbf{X}_2$ are vectors and C_1 , C_2 are scalars is called a *linear combination* of $\mathbf{X}_1$, $\mathbf{X}_2$ with coefficients C_1, C_2.

17. Given the following system

$$X' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} X$$

Show that the vector functions

$$X_1 = \begin{pmatrix} 2\cos t \\ -\cos t + \sin t \\ -2\cos t - 2\sin t \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}$$

are solutions of the given system. Discuss a linear combination of these solutions.

18. By Superposition Principle, if n vector functions

$$\mathbf{X}_1(t) = \begin{pmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \dots, \mathbf{X}_n(t) = \begin{pmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}$$

are solutions of the homogeneous system (2), then the linear combination

$$\mathbf{X}(t) = C_1\mathbf{X}_1(t) + \ldots + C_n\mathbf{X}_n(t) \tag{3}$$

is also a solution of (2) for any constants $C_1, \dots, C_n$.

19. Consider the $n \times n$ matrix, whose columns are vectors $X_1(t), \dots, X_n(t)$:

$$\Psi(t) = \begin{pmatrix} x_{11}(t) & \dots & x_{1n}(t) \\ \vdots & \vdots & \vdots \\ x_{n1}(t) & \dots & x_{nn}(t) \end{pmatrix}. \tag{4}$$

Question: Under what conditions on vector functions $\mathbf{X}_1(t), \dots \mathbf{X}_n(t)$ the right-hand side of (3) constitute the general solution of (2)? If this the case one says that the set of vector functions $\mathbf{X}_1(t), \dots \mathbf{X}_n(t)$ form a *fundamental set of solutions* of system (2).

THEOREM 1. *Equation* (3) *give the general solutions of linear homogeneous system* (2) *if and only if the* **determinant** *of the matrix* $\Psi(t)$ *from* (4) *is not equal to zero for some time moment* $t = t_0$.

You already met and used determinants of 2×2 and 3×3 matrices , at least when you studied vector and scalar triple product in your Calc 3. Next time I will introduce the determinant of $n \times n$ matrix, discuss its basic properties, what does this determinant actually determines, and why Theorem 1 is valid.