11. Determinants and fundamental set of solutions of linear homogeneous first order systems of ODEs (section 7.4)

Determinant

1. Determinant of a matrix is a function that takes a square matrix and converts it into a number.

2. Determinant of 2×2 and 3×3 matrices.

   - A **determinant of order 2** is defined by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

   - A **determinant of order 3** is defined by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

3. **determinant of** 4×4 **and in general** $n\times n$ **matrices through the inductive procedure of the expansion w.r.t the first row**

4. In the same way one can expand with respect any row and any column according to the following chess-board rule:

5. **determinant of $n \times n$ matrices-closed form formula using permutations (can be skipped).**

   - Case $n = 2$. Below each term of the expansion write the second indices in the order in which they appear in the term (assuming that the factors of each term are arranged such that the first indices appear in the natural order):

   $$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \underbrace{a_{11}a_{22}}_{1,2)} - \underbrace{a_{12}a_{21}}_{2,1}$$

   Observation: We have 2 terms in this expansion and each term corresponds to a permutation of the set $\{1, 2\}$ and appears with certain sign, which depends on this permutation.

   - Case $n = 3$

   $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underbrace{a_{11}a_{22}a_{33}}_{1,2,3} - \underbrace{a_{11}a_{23}a_{32}}_{1,3,2} - \underbrace{a_{12}a_{21}a_{33}}_{2,1,3} + \underbrace{a_{12}a_{23}a_{31}}_{2,3,1} + \underbrace{a_{13}a_{21}a_{32}}_{3,1,2} - \underbrace{a_{13}a_{22}a_{31}}_{3,2,1}$$

   Observation: We have $6 = 3!$ terms in this expansion and each term corresponds to a permutation of the set $\{1, 2, 3\}$ and appears with certain sign, which depends on this permutation.

   - General n (just an idea, no details): A permutation of n elements is described by a map σ from the set $\{1, \dots, n\}$ to itself which is one-to-one and onto (which in this case is equivalent to the fact that in $(\sigma(1), \dots, \sigma(n))$ there is no repetitions). To any permutation σ one can assign a signature $\text{sgn}(\sigma)$ which is either $+$ or $-$ according to the following nice combinatorial rule:

   Then the general formula for the determinant of $n \times n$ matrix is

   $$\det A = \sum_{\sigma \text{is a permutation of} \{1,\dots,n\}} \text{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}.$$

6. The latter formula is not efficient in computation of determinants but it is very useful for understanding general properties of the determinant.

   The following properties of determinants are very useful for their calculations:

(a) If one multiplies a row (a column) of a matrix by a constant, then the determinant of the resulting matrix is equal to the determinant of the original matric times this constant;

(b) If a row (or a column) is a sum of two row vectors (two column vectors) **a** and **b**, then the determinant of the matrix is equal to the sum of the determinants of two matrices obtained from the original matrix by replacing the same row/ column first **a** and then by **b**;

(c) Switching two rows/ columns in a matrix results in changing the sign of the determinant.

As a consequence, if a matrix have two identical rows/olumns its determinant is equal to zero.

(d) (a consequence of the previous two items but the most important tool in calculating the determinannts) The determinant of a matrix is not changed if one subtract a row multiplied by a constant from another row. The same is true for columns.

What does determinant determine?

THEOREM 1. *Let A be $n \times n$ matrix. The following three conditions are equivalent:*

(a) The linear algebraic system

$$A\mathbf{X} = b$$

has a unique solution for any n-dimensional column vector b;

(b) $\det A \neq 0$;

(c) The linear algebraic homogeneous system

$$A\mathbf{X} = \mathbf{0}$$

has a unique solution $\mathbf{X} = \mathbf{0}$ (called the trivial *solution).*

Proof in the case of $n = 2$

7. Returning to last page of the previous set of notes:

8. We consider a system of n first order linear homogeneous ODEs:

$$\mathbf{X}' = P(t)\mathbf{X}. \tag{1}$$

9. Assume that n vector functions

$$\mathbf{X}_1(t) = \begin{pmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \dots, \mathbf{X}_n(t) = \begin{pmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}$$

are solutions of the homogeneous system (1) and consider the $n \times n$ matrix, whose columns are vectors $X_1(t), \dots, X_n(t)$:

$$\Psi(t) = \begin{pmatrix} x_{11}(t) & \dots & x_{1n}(t) \\ \vdots & \vdots & \vdots \\ x_{n1}(t) & \dots & x_{nn}(t) \end{pmatrix}. \tag{2}$$

10. The determinant of the matrix $\Psi(t)$ is called the **Wronskian** of the solutions $X_1, \dots, X_n$ and is denoted by

$$W\left[X_1, \dots, X_n\right](t) = \det \Psi(t).$$

THEOREM 2. *The general solutions of linear homogeneous system* (1) *is given by*

$$X(t) = C_1 X_1(t) + \ldots + C_n X_n(t) \tag{3}$$

if and only if the Wronskian $W\left[X_1, \ldots, X_n\right](t) = \det \Psi(t)$ *of the solutions* $X_1, \ldots, X_n$ *is not equal to zero for some time momemnt* $t = t_0$.

Proof

11. Note that $det\Psi(t_0) \neq 0$ implies $det\Psi(t) \neq 0$ for any t.

EXPLANATION

12. If the Wronskian of the solutions $X_1, \ldots, X_n$ is not zero at some (and therefoe any) time moment, then $X_1, \ldots, X_n$ is called the **fundamental set of solutions** and the general solution of the system (1) is $C_1X_1(t) + \ldots + C_nX_n(t)$.

13. Given that the vector functions $X_1 = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$ and $X_2 = \begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}$ are solutions of the system $X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$. Find general solution of these system.

14. Question: *How to find a fundamental set of solutions?*

    In the next section we answer it for the case $P(t) = A$, i.e. for system of linear homogeneous equations with constant coefficients (section 7.5) and apply it to second order equation (sections 3.1 and 3.2).