12. System of homogeneous linear equations with constant coefficients: the role of eigenvalues and eigenvetors, the case of distinct eigenvalues(sec 7.3 and 7.5)

1. A number λ is called an **eigenvalue** of matrix A if there exists a **nonzero** vector $\mathbf{v}$ such that
$$A\mathbf{v} = \lambda\mathbf{v},$$
and $\mathbf{v}$ is called an **eigenvector** corresponding to the eigenvalue λ.

2. Example (corresponds to uncoupled systems). If A is a *diagonal matrix*,
$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \qquad (*)$$
then the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues and the vectors
$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \quad \mathbf{v_n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$
are the corresponding eigenvectors.

3. Relation of diagonal matrices to uncoupled systems:

The system corresponding to the diagonal matrix (*) is uncoupled*

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 \Rightarrow x_1(t) = C_1 e^{\lambda_1 t} \\ \dot{x}_2 = \lambda_2 x_2 \Rightarrow x_2(t) = C_2 e^{\lambda_2 t} \\ \vdots \\ \dot{x}_n = \lambda x_n \Rightarrow x_n(t) = C_n e^{\lambda_n t} \end{cases} \Rightarrow$$

The general solution is
$$X(t) = \begin{pmatrix} C_1 e^{\lambda_1 t} \\ \vdots \\ C_n e^{\lambda_n t} \end{pmatrix} = C_1 e^{\overset{\text{eigenvalue}}{\lambda_1} t} \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\text{eigenvector}} + C_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + C_n e^{\lambda_n t}$$

* Uncoupled means that each equation contains exactly one unknown function

4. **Fundamental Proposition.** *If λ is an eigenvalue of matrix A and $\mathbf{v}$ is an eigenvector corresponding to this eigenvalue then*
$$\mathbf{X}(t) = e^{\lambda t}\mathbf{v}$$
is a solution of the system $\mathbf{X}' = A\mathbf{X}$, i.e solution of the homogeneous linear system with constant coefficients.

<u>Proof</u> Plug $X(t) = e^{\lambda t} v$ into $X' = AX$

L.H.S. $X'(t) = \frac{d}{dt}(e^{\lambda t} \underbrace{v}_{\text{Constant vector}}) = \frac{d}{dt}(e^{\lambda t}) v = \lambda e^{\lambda t} v;$

R.H.S. $AX(t) = A(e^{\lambda t} v) = e^{\lambda t} \underbrace{Av}_{\lambda v} = \lambda e^{\lambda t} v$

So $X'(t) = AX(t)$, i.e. $X(t) = e^{\lambda t} v$ is a solution of $X' = AX$

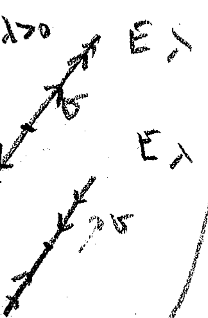
5. Geometric interpretation of Fundamental Proposition.

Let $\vec{F}(X) = AX$. This is the vector field corresponding to the system $X' = AX$. Let E_λ be the set of vectors parallel to v: $E_\lambda = \{cv : c \in \mathbb{R}\}$

...ther words, ... is ... the ...erated ... v

Then if $X \in E_\lambda$ then $\vec{F}(X) = AX \parallel v$ (indeed $AX = Acv = cAv = c\lambda v$)

In other words, for any $X \in E_\lambda$ the vector $\vec{F}(X)$ is tangent to $E_\lambda \Rightarrow$ If $X(t)$ is a solution of $X' = AX$ and $X(0) \in E_\lambda$ then $X(t) \in E_\lambda$ for any $t \Rightarrow X(t) = \alpha(t) v$ for some function $\alpha(t)$. Plug this into the system: $X'(t) = \alpha'(t) v$ and $AX(t) = A\alpha(t) v = \alpha(t) Av = \lambda \alpha(t) v$ $\Rightarrow \alpha'(t) = \lambda \alpha(t) \Rightarrow$ to find $\alpha(t)$: $\alpha(t) = ce^{\lambda t} \Rightarrow X(t) = ce^{\lambda t} v$ (so $e^{\lambda t} v$ is a solution)



6. Eigenvalue are solutions of the following **characteristic equation** (**roots of the following characteristic polynomial**):

$$\det(A - \lambda I) = 0.$$

What is the degree of this polynomial?

λ is an eigenvalue of A if there exist $v \neq 0$ s.t. $Av = \lambda v$

$v = \underbrace{I}_{\text{the identity matrix}} v \Rightarrow$ $Av = \lambda v \Leftrightarrow Av = \lambda I v \Leftrightarrow Av - \lambda I v = 0 \Leftrightarrow$ $(A - \lambda I) v = 0$, i.e. λ is an eigenvalue if and only if the linear homogeneous system $(A - \lambda I) v = 0$ has a nontrivial solution $\Leftrightarrow$ (see Theorem 1 of section 7.?) $\det(A - \lambda I) = 0 \to$ $\boxed{\text{characteristic equation}}$

7. **Trace of an** $n \times n$ **matrix** A is the sum of it diagonal elements, denoted by trace(A) or tr(A):

$$\det(A - \lambda I) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{pmatrix} = \text{a polynomial of } \boxed{\text{degree } n} \to$$

characteristic polynomial

has exactly n (complex) roots (counting multiplicities)

$$\mathrm{trace}(A) = a_{11} + a_{22} + \dots a_{nn}.$$

8. Show that the characteristic equation in the case $n = 2$ can be found as

$$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A) = 0.$$

$$\det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} =$$

$$= a_{11}a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2 - a_{12}a_{21} = \lambda^2 - \underbrace{(a_{11}+a_{22})}_{\mathrm{tr} A}\lambda + \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{\det A}$$

Note that the fact that the term without λ (the free term) in the characteristic polynomial is equal to $\det A$ follows from the general fact that this term is equal to the value of the polynomial at $\lambda = 0$

Plugging $\lambda = 0$ into $\det(A - \lambda I)$ we get $\det A$

9. Consequently to find eigenvalues of an 2×2 matric we need to solve a quadratic equation (and more generally, to find eigenvalues of $n \times n$ matrix we need to find roots of a polynomial of degree n).

10. `Fact from Algebra:` The quadratic equation $a\lambda^2 + b\lambda + c = 0$ has roots

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

which fall into one of 3 cases:

- two distinct real roots $\lambda_1 \neq \lambda_2$ (in this case $D = b^2 - 4ac > 0$) [corresponds to part of section 7.5 and can be applied to section 3.1
- two complex conjugate roots $\lambda_1 = \overline{\lambda_2}$ (in this case $D = b^2 - 4ac < 0$) [corresponds to a part of section 7.6 and can be applied to section 3.3]
- two equal real roots $\lambda_1 = \lambda_2$ (in this case $D = b^2 - 4ac = 0$) [[corresponds to a part of section 7.8 and can be applied to section 3.4]

Real Distinct Eigenvalues

11. *FACT from Linear algebra: If A has distinct eigenvalues $\lambda_1, \dots, \lambda_n$ and $\mathbf{v}^1, \dots, \mathbf{v}^n$ are the corresponding eigenvectors, then $\det(v^1, \dots, v^n)$ (i.e. the determinant of the matrix with jth column equal to $\mathbf{v}^j$) does not vanish or, equivalently the collection of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ form a basis of $\mathbb{R}^n$*

As a consequence, if If A has <u>distinct real</u> eigenvalues $\lambda_1, \dots, \lambda_n$ with eigenvectors $\mathbf{v}^1, \dots, \mathbf{v}^n$, then

$$\{e^{\lambda_1 t}\mathbf{v}^1, \dots, e^{\lambda_n t}\mathbf{v}^n\}$$

is a fundamental set of solutions and the general solution is

$$\mathbf{X}(t) = C_1 e^{\lambda_1 t}\mathbf{v}^1 + \dots + C_n e^{\lambda_n t}\mathbf{v}^n. \tag{1}$$

Indeed, the Wronskian of the set of solutions $\{e^{\lambda_1 t}v^1, \dots, e^{\lambda_n t}v^n\}$ at $t=0$ is equal to $\det(v^1, \dots, v^n)$ and it is not equal to zero $\Rightarrow$ (1) is the general solution by Theorem 2 of section 11.

REMARK If the eigenvalues are distinct but some of them are complex, the formula (1) gives the general complex-valued solutions, so we need to make additional work to get the general real-valued solutions (will be discussed next week and corresponds to section 7.6 and 3.3, we also will make a thorough review of complex numbers there).

12. EXAMPLE. Consider the following system of ODEs:

$$\begin{aligned} x_1' &= -2x_1 + x_2 \\ x_2' &= 2x_1 - 3x_2 \end{aligned} \tag{2}$$

(a) Find general solution of (2).

$A = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}$ • Find the eigenvalues

Characteristic equation: $\operatorname{tr} A = -5$, $\det A = 6-2=4$

$\Rightarrow$ the characteristic equation is $\lambda^2 + 5\lambda + 4 = 0$

$D = 25 - 4\cdot 4 = 9$

$\underbrace{\lambda_1 = \frac{-5+3}{2} = -1, \quad \lambda_2 = \frac{-5-3}{2} = -4}_{\text{distinct real}}$

• An eigenvector for $\lambda = -1$

Solve $(A - (-1)I)v = 0$

$(A-(-1)I)v = (A+I)v = \begin{pmatrix} -2+1 & 1 \\ 2 & -3+1 \end{pmatrix} v = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow -v_1 + v_2 = 0$. Take $v_2 = 1 \Rightarrow v_1 = 1 \Rightarrow v^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = -1$

(Rem: the second equation is a multiple of the first! This is always the case when one search for eigenvalues in the case of $n=2$)

An eigenvector of $\lambda = -4$

$(A-(-4)I)v = (A+4I)v = \begin{pmatrix} -2+4 & 1 \\ 2 & -3+4 \end{pmatrix} v = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\Leftrightarrow 2v_1 + v_2 = 0$. Take $v_1 = -1 \Rightarrow v_2 = 2 \Rightarrow v^2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is an eigenvector of $\lambda = -4$

general solution

$\boxed{X(t) = C_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-4t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}}$

(b) Find solution of (2) subject to the initial condition $X(0) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

Plug $t=0$ into general solution

$$X(0) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \Rightarrow$$

$C_1 - C_2 = 1$ (Eq 1) Eliminate C_1 : (Eq 2) - (Eq 1):

$C_1 + 2C_2 = 4$ (Eq 2) $3C_2 = 3 \Rightarrow \boxed{C_2 = 1} \Rightarrow C_1 = 1 + C_2 = 2$

$$\Rightarrow \boxed{X(t) = 2e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-4t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}}$$

(c) What is behavior of the solution as $t \to +\infty$?

Since $e^{-4t} \xrightarrow[t \to +\infty]{} 0$, $e^{-t} \xrightarrow[t \to +\infty]{} 0$

$$\boxed{\lim_{t \to +\infty} X(t) = 0}$$ (independently of initial conditions)

13. Applications to second and higher order linear homogeneous equations (sections 3.2, 3.1 for second order, 4.1, 4.2 for higher order, the latter can be skipped)

1. Consider a linear homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0 \tag{3}$$

with coefficients p and q being continuous in an interval I. Then, as was already discussed in section 9 item 7 of the notes, this equation can be transformed to the following system of first order equations, by setting $x_1(t) := y(t)$, $x_2(t) = y'(t)$:

$$\begin{cases} x_1' = x_2 \\ x_2' = -q(t)x_1 - p(t)x_2. \end{cases} \tag{4}$$

so that a function $y(t)$ is a solution of (3) if and only of the vector function

$$\mathbf{X}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

is a solution of (4).

Then as a consequence of general theory for systems (section 10, items 16-18, and section 11, items 8-12) we get

2. **Superposition Principle for second (and also higher) order equations** reads: If $y_1(t)$ and $y_2(t)$ are two solutions of (3), then

$$y(t) = C_1 y_1(t) + C_2 y_2(t). \tag{5}$$

is a solution of (3).

Indeed by superposition principle for system (section 10) $C_1 \begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix} + C_2 \begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} C_1 y_1(t) + C_2 y_2(t) \\ C_1 y_1'(t) + C_2 y_2'(t) \end{pmatrix}$ is a solution of (4) $\Rightarrow$ $C_1 y_1(t) + C_2 y_2(t)$ is a solution of (3)

3. **Wronskian of two solutions** Take two solutions $y_1(t)$ and $y_2(t)$ of (3). Then then

$$\mathbf{X}_1(t) = \begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix}, \mathbf{X}_2(t) = \begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix}$$

are two solutions of (4). Therefore the 2×2 matrix $\Psi(t)$ formed from this two solutions is

$$\Psi(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \tag{6}$$

DEFINITION 1. *The determinant of the matrix $\Psi(t)$ is called* **WRONSKIAN** *of the functions $y_1(t)$ and $y_2(t)$ and it is denoted by $W(y_1, y_2)(t)$:*

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

4. As a consequence of Theorem 2 of section 10 we get

   THEOREM 2. *Let $y_1(t)$ and y_2 are solutions of (3). The general solution of (3) is given by $y(t) = C_1y_1(t) + C_2y_2(t)$ if and only if the Wronskian $W(y_1, y_2)(t_0) \neq 0$ for some time moment t_0.*

5. As discussed in the very first lecture $\cos t$ and $\sin t$ are solutions of $y'' + y = 0$. Using the developed theory, justify that $y(t) = C_1 \cos t + C_2 \sin t$ is the general solution of this equation

Check if $W(\cos t, \sin t)$ is not zero

$$W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ (\cos t)' & (\sin t)' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} =$$

$\cos^2 t + \sin^2 t = 1 \neq 0 \Rightarrow y(t) = C_1 \cos t + C_2 \sin t$ is the general solution

6. **Generalization for equation of order n (chapter 4)** What is the Wronskian for n solutions of an equation of nth order and the analog of Theorem (2) in this case?

Linear homogeneous equation of order n:

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \dots + a_1(t)y'(t) + a_0(t)y(t) = 0 \quad (*)$$

$$\begin{matrix} x_1 = y(t) \\ x_2 = y'(t) \\ \vdots \\ x_n = y^{(n-1)}(t) \end{matrix} \Rightarrow \begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = x_3(t) \\ \vdots \\ x_n'(t) = -a_0(t)x_1(t) - a_1(t)x_2(t) - \dots - a_{n-1}(t)x_n(t) \end{cases} \quad (**)$$

$y(t)$ is a solution of $(*) \Leftrightarrow X(t) = \begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix}$ is a solution of $(**) \Rightarrow$

The Wronskian of n solutions $y_1(t), \dots, y_n(t)$ of $(*)$ is

$$W(y_1, \dots, y_n)(t) = \begin{vmatrix} y_1(t) & \dots & y_n(t) \\ y_1'(t) & \dots & y_n'(t) \\ \vdots & & \vdots \\ y_1^{(n-1)}(t) & \dots & y_n^{(n-1)}(t) \end{vmatrix} \text{ and}$$

$y(t) = C_1y_1(t) + \dots + C_ny_n(t)$ is the general solution of $(*) \Leftrightarrow W(y_1, \dots, y_n)(t) \neq 0$ for some $t = t_0$

The case of linear homogeneous equations of second order: characteristic equation and general solution in the the case of real distinct roots (sec. 3.1)

7. Consider

$$ay'' + by' + cy = 0 \tag{7}$$

with constant real coefficients a, b, and c, $a \neq 0$. The corresponding system of first order equation is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\tfrac{c}{a}x_1 - \tfrac{b}{a}x_2. \end{aligned} \tag{8}$$

Indeed $ay'' + by' + cy = 0 \Leftrightarrow y'' = -\frac{c}{a}y - \frac{b}{a}y'$

$x_1 = y$, $x_2 = y'$ $\Rightarrow$ $x_1' = x_2$, $x_2' = -\frac{c}{a}x_1 - \frac{b}{a}x_2$

8. Show that the characteristic equation for the eigenvalues of the matrix of the system (8) is equivalent to

$$a\lambda^2 + b\lambda + c = 0. \tag{9}$$

Indeed, the matrix corresponding to system 8 is

$A = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \Rightarrow \operatorname{tr} A = -\frac{b}{a}$, $\det A = \frac{c}{a} \Rightarrow$

The characteristic equation of A is $\lambda^2 - \operatorname{tr} A\,\lambda + \det A = 0$

$\Leftrightarrow \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} \Leftrightarrow a\lambda^2 + b\lambda + c = 0$

Note that the characteristic equation (9) can be determined from the original second differential equation (7) simply by replacing $y^{(k)}$ with λ^k (you relate to y itself as to the derivative of y of order 0).

The case of two distinct real roots λ_1 and λ_2 of (9) $\Leftrightarrow$ distinct real eigenvalues of the matrix of the corresponding system (8). Therefore the general solution of (7) is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2} t. \tag{10}$$

EXPLANATION

Let v^1 is an eigenvector of λ_1

Note that if $v^1 = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$ then $v_{11} \neq 0$

(Indeed $\begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} v_{12} \\ * \end{pmatrix} = \lambda_1 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$, i.e.

$v_{12} = \lambda_1 v_{11}$ $\Rightarrow$ If $v_{11} = 0$ then $v_{12} = 0 \Rightarrow v^1 = 0$, which contradicts the definition of an eigenvector

So $v_{11} \neq 0$ $\Rightarrow$ we can assume that $v_{11} = 1$ $\Rightarrow$ $\begin{pmatrix} e^{\lambda_1 t} \\ \lambda_1 e^{\lambda_1 t} \end{pmatrix}$ is a solution of (8) $\Rightarrow$ $y(t) = e^{\lambda_1 t}$ is a solution of (7).

In the same way $e^{\lambda_2 t}$ is a solution of 7 $\Rightarrow$ $C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$ is the general solution

$W(e^{\lambda_1 t}, e^{\lambda_2 t})(0) = \lambda_2 - \lambda_1 \neq 0$)

9. More elementary derivation of (9) and (10) without using the notions of eigenvalues and eigenvectors: the nature of the equation (7) suggests that it may have solutions of the form$y = e^{\lambda t}$. Plug it to (7):

$c \times \quad y(t) = e^{\lambda t}$

$b \times \quad y'(t) = \lambda e^{\lambda t}$

$a \times \quad y''(t) = \lambda^2 e^{\lambda t}$

$a y''(t) + b y'(t) + c y = (a\lambda^2 + b\lambda + c) e^{\lambda t} = 0$

$\Rightarrow \boxed{a\lambda^2 + b\lambda + c = 0}$

(no eigenvalues / eigenvectors were used)

10. EXAMPLE. Consider

$$3y'' - y' - 2y = 0.$$

(a) Find general solution.

Characteristic equation: $3\lambda^2 - \lambda - 2 = 0$

$D = 1 - 4 \cdot 3 \cdot (-2) = 25$

$\lambda_1 = \frac{1+5}{6} = 1$, $\lambda_2 = \frac{1-5}{6} = -\frac{2}{3}$ } distinct real

$\Rightarrow$ General solution is $y(t) = C_1 e^t + C_2 e^{-2/3 t}$

(So for solving this type of problem you do not need to refer to eigenvalues and eigenvectors, although implicitly they are there)

(b) Find solution satisfying the following initial conditions: $y(0) = \alpha$, $y'(0) = 1$, where α is a real parameter.

$y(0) = \alpha \Rightarrow C_1 + C_2 = \alpha$

$y'(t) = C_1 e^t - \frac{2}{3} C_2 e^{-2/3 t}$

$y'(0) = 1 \Rightarrow C_1 - \frac{2}{3} C_2 = 1$

So we have a system of 2 linear equations for C_1 & C_2:

$\begin{cases} C_1 + C_2 = \alpha & (\text{Eq } 1) \\ C_1 - \frac{2}{3} C_2 = 1 & (\text{Eq } 2) \end{cases}$

Eliminate C_1: (Eq 1) $-$ (Eq 2):

$C_2 - \left(-\frac{2}{3}\right) C_2 = \alpha - 1 \Rightarrow$

$\Rightarrow \frac{5}{3} C_2 = \alpha - 1 \Rightarrow C_2 = \frac{3}{5}(\alpha - 1)$. From (Eq 1) $C_1 = \alpha - C_2 = \alpha - \frac{3}{5}(\alpha - 1) = \frac{3}{5} + \frac{2}{5}\alpha = \frac{1}{5}(2\alpha + 3)$

So $\boxed{y(t) = \frac{1}{5}(2\alpha + 3) e^t + \frac{3}{5}(\alpha - 1) e^{-2/3 t}}$

(c) Find all α so that the solution of the corresponding IVP approaches 0 as $t \to +\infty$.

$e^{-2/3 t} \underset{t \to +\infty}{\longrightarrow} 0$, $e^t \underset{t \to +\infty}{\longrightarrow} +\infty$ so

$y(t) = \frac{1}{5}(2\alpha + 3) e^t + \frac{3}{5}(\alpha - 1) e^{-2/3 t} \underset{t \to +\infty}{\longrightarrow} 0 \iff 2\alpha + 3 = 0 \Rightarrow \boxed{\alpha = -\frac{3}{2}}$

11. How to generalize this theory to linear equation with constant coefficients of order n?

Given an equation $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$ (10)

where $a_n \neq 0$, $a_1, \ldots, a_n$ are constant, consider the polynomial equation $a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0$ (the characteristic equation). If it has n distinct real solutions $\lambda_1, \lambda_2, \ldots, \lambda_n$ then the general solution of (10) is $\boxed{C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \ldots + C_n e^{\lambda_n t}}$