14. System of homogeneous linear equations with constant coefficients: example for $n = 3$ and practice of the Gauss elimination method for funding of eigenvectors and solving initial value problems (section 7.2 and 7.5)

Example. Given

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

1. Find eigenvalues of A.

   REMARK 1. *Here the characteristic equation is cubic. I do not expect you to use the Cardano formula for cubic equation. In the problems you will be given the roots are usually rational. We will use the following rule:*

   If $\lambda = \dfrac{p}{q}$ *is a rational number written in the lowest terms (i.e. p nd q are coprime, i.e. having the greatest common factor 1) and it is a root of a polynomial equation*

   $$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$$

   with integer coefficients a_0, a_1, $\ldots a_n$, then p must divide a_0 and q must divide a_n.

2. Find eigenvectors of A (use Gauss Elimination Method below).

Gauss Elimination Method

(a) The Gauss Method is a suitable technique for solving systems of linear equations of any size. A sequence of operations (see below) of the Gauss-Jordan elimination method allows us to obtain at each step an equivalent system - that is, a system having the same solution as the original system.

The operations of the Gauss-Jordan elimination method are

i. Interchange any two equations.

ii. Replace an equation by a nonzero multiple of itself.

iii. Replace an equation by itself plus a nonzero multiple of any other equation.

(b) An **augmented matrix** that is formed by combining the coefficient matrix and the constant matrix. For example, for the system of linear equations $\begin{cases} 3x_1 & + & 12x_2 & = & 20 \\ 2x_2 & = & x_1 & +7 & \end{cases}$ the augmented matrix is $\left(\begin{array}{cc|c} 3 & 12 & 20 \\ 1 & -2 & -7 \end{array}\right)$.

(c) The goal of the Gauss Elimination Method is to get the augmented matrix into **Reduced Echelon Form**. A matrix is in row echelon form if

- All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes (all zero rows, if any, belong at the bottom of the matrix).
- The leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.
- All entries in a column below a leading entry are zeroes (follows from the first two criteria).

(d) To put a matrix in Reduced Form, there are three valid

Row Operations:

i. Interchange any two rows ($R_i \leftrightarrow R_j$).
ii. Replace any row by a nonzero constant multiple of itself ($R_i \leftrightarrow cR_i$).
iii. Replace any row by the sum of that row and a constant multiple of any other row $R_i \leftrightarrow (R_i + cR_j)$.

Each of this operation corresponds to an operation on the original system which does not change the set of solutions of the system (i.e. we obtain an equivalent system to the original one, having exactly the same set of solutions)

Application of this method for finding eigenvectors of A:

3. Find the general solution of the system of differential equation

$$\begin{aligned} x_1' &= x_1 - x_2 + 4x_3 \\ x_2' &= 3x_1 + 2x_2 - x_3 \\ x_3' &= 2x_1 + x_2 - x_3 \end{aligned} \tag{1}$$

4. Find the solution of system (1) satisfying the intiial conditions $x_1(0) = 0, x_2(0) = 11, x_3(0) = 4$.