15: The case of complex eigenvalues/roots of the characteristic equation (sections 3.3 and 7.6 combined).

Review of complex numbers

1. Equation $\lambda^2 + 1 = 0$ doesn't have real roots! What to do? Introduce a new number i, the *imaginary unit*, such that $i^2 = -1$. So, $\lambda^2 + 1 = 0$ implies that $\lambda = \pm\sqrt{-1} = \pm i$.

2. A complex number z is a pair of two real numbers x and y or geometrically a point (x, y) in the plane written in the form $x + iy$.

   Draw the following complex numbers on the plane

   (a) i

   (b) $-1 - i$

3. What is more important one can define the addition and multiplication of any two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ can be defined in the following natural way:

   (a) **Addition**

   $$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

   Geometrically, it is the addition of the correspondong position vectors in the plane

   (b) **Multiplication**: just use the distributive law and the fact that $i^2 = -1$ to define

   $z_1 z_2 =$

   EXAMPLE 1. Calculate $(1 + i)^2$, i.e. represent it in the form $a + ib$

4. (a) **Real part** of $z = x + iy$: $\mathrm{Re}z := x$.

   (b) **Imaginary part** of $z = x + iy$: $\mathrm{Im}z := y$.

   (c) **Complex conjugate** of $z = x + iy$ is the number $\bar{z} := x - iy$.

   (d) Real and imaginary parts in terms of complex conjugates:

   $$\mathrm{Re}z = \frac{z + \bar{z}}{2}, \quad \mathrm{Im}z = \frac{z - \bar{z}}{2i}. \tag{1}$$

   (e) $z\bar{z} = x^2 + y^2$.

   EXAMPLE 2. **How to divide complex numbers:** Calculate $\dfrac{5 + 4i}{3 + 2i}$

5. **Complex numbers in polar coordinates**

   Let (r, θ) are polar coordinates: $x = r\cos\theta, y = r\sin\theta$. Then

   $$z = x + iy = r(\cos\theta + i\sin\theta)$$

   The common terminology:

   $|z| = r = \sqrt{x^2 + y^2}$ is called the *modulus* of complex numbers.

   $\arg z = \theta$ is called the *argument* of the complex number z (it is not defined uniquely but up to $2\pi k$, where k is an integer).

   EXAMPLE 3. Given a complex number z find $|z|$ and $\arg z$ (for the latter we are interested in the value in the interval $[0, 2\pi)$).

   (a) z=i

   (b) z=1+i

6. **Multiplication of complex numbers in polar coordinates**

   Reminder from trigonometry:

   $$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \tag{2}$$

   $$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \tag{3}$$

   Proof of (2): equivalence of two definition of the dot product:

   Take two vectors

   $$\mathbf{a} = \langle\cos\theta_1, \sin\theta_1\rangle, \quad \mathbf{b} = \langle cos(-\theta_2), \sin(-\theta_2)\rangle = \langle\cos(\theta_2), -\sin(\theta_2)\rangle.$$

   Then, on one hand, $|\mathbf{a}| = |\mathbf{b}| = 1$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is equal to $\theta_1 + \theta_2$. Therefore,

   $$\mathbf{a}\cdot\mathbf{b} = \cos(\theta_1 + \theta_2).$$

   On the other hand, using the formula for the dot product via components:

   $$\mathbf{a}\cdot\mathbf{b} = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2. \quad \square$$

7. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

   Then

   $z_1 z_2 =$

Conclusion

$$|z_1 z_2| = |z_1||z_2|$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2. \tag{4}$$

EXAMPLE 1 (revisited): Calculate $(1+i)^2$ using polar coordinates

8. Preliminary of the Euler formula

   If $f(\theta) := \cos\theta + i\sin\theta$, then by (4)

   $$f(\theta_1)f(\theta_2) = f(\theta_1 + \theta_2).$$

   So, $f\theta)$ behaves like an exponential function (see the Euler formula (5) below for more exact statement).

9. **The complex exponential via Taylor Expansion.**

   (a) Recall that the Taylor expansion of e^x is

   (b) Nothing prevent us from replacing real x by a complex z in this formula to define e^z:

   However we need a better definition of exponential not using series.

10. The Euler formula and the exponent of the complex number via the Euler formula.

    $$\boxed{e^{iy} = \cos y + i\sin y} \tag{5}$$

    EXPLANATION: Taylor expansions of

    (a) $\cos y =$

    (b) $\sin y =$

    (c) $e^{iy} =$

    $1, i, i^2 = \quad, i^3 = \quad, i^4 = \quad \dots$

    □

    Therefore,

    $e^{x+iy} =$

    CONCLUSION:

    $$\boxed{e^{x+iy} = e^x(\cos y + i\sin y)} \tag{6}$$

$$\operatorname{Re}\left(e^{x+iy}\right) = e^x \cos y, \quad \operatorname{Im}\left(e^{x+iy}\right) = e^x \sin y. \tag{7}$$

11. Similarly to a real r for any complex r

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}. \tag{8}$$

   (a) One way to prove this identity is to use term by term differentiation of the Taylor series for e^{rt}:

   REMARK 4. *The great thing in this method is that it works exactly in the same way for the exponential of matrices and therefore gives similar result for systems of first order equation as we will see studying chapter 7*

   (b) Another way to prove (8) is to use (6) : if $r = \alpha + i\omega$. Then by (6)

$$\boxed{e^{\lambda t} = e^{\alpha t}\Big(\cos(\omega t) + i\sin(\omega t)\Big)}. \tag{9}$$

   Differentiate this identity to get (8) (try!, differentiate the real and the imaginary part).

Second order homogeneous equations with constant coefficients: the case of two complex conjugate roots $\lambda_1 = \overline{\lambda_2}$ for the characteristic equation(in this case $D = b^2 - 4ac < 0$)

12. Now we return to a second order linear homogeneous equation with constant real coefficients

$$ay'' + by' + cy = 0 \tag{10}$$

Recall that the **characteristic equation** of (10) is

$$a\lambda^2 + b\lambda + c = 0. \tag{11}$$

We consider the case

$$D = b^2 - 4ac < 0.$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = -\frac{b}{2a} \pm i\frac{\sqrt{|D|}}{2a} =: \alpha \pm i\omega$$

Note that $\lambda_2 = \bar{\lambda}_1$. From now one let $\lambda = \lambda_1$.

13. Since in this case $r \neq \bar{r}$, exactly as in the case of distinct real roots two particular solutions

$$y_1 = e^{\lambda t}, \qquad y_2 = e^{\bar{\lambda} t}.$$

form a fundamental set of solutions.

However, **this are complex-valued solutions** and we are interested in the real valued one. So the answer for the general solution $C_1 e^{(\alpha+i\omega)t} + C_2 e^{(\alpha-i\omega)t}$ is formally true but **not acceptable** in this course.

14. Note that

$$e^{\lambda t} = e^{\alpha t}\Big(\cos(\omega t) + i\sin(\omega t)\Big) \quad e^{\bar{\lambda} t} = e^{\alpha t}\Big(\cos(\omega t) - i\sin(\omega t)\Big)$$

and so $e^{\bar{\lambda} t} = \overline{e^{\lambda t}}$).

By Superposition Principle from the previous formulas (see also formula (1) above) the following linear combinations are solutions as well

$$\mathrm{Re}\left(e^{\lambda t}\right) = \tfrac{1}{2}(e^{\lambda t} + e^{\bar{\lambda} t}) = e^{\alpha t}\cos(\omega t), \quad \mathrm{Im}\left(e^{\lambda t}\right) = \frac{1}{2i}(e^{\lambda t} - e^{\bar{\lambda} t}) = e^{\alpha t}\sin(\omega t) \quad (12)$$

Note that these solutions are real-valued functions.

15. Solutions $\{e^{\alpha t}\cos(\omega t), e^{\alpha t}\sin(\omega t)\}$ is a fundamental set of solutions, i.e. that general solution of (10) has a form

$$\boxed{y(t) = C_1 e^{\alpha t}\cos(\omega t) + C_2 e^{\alpha t}\sin(\omega t)} \quad (13)$$

This follows from the calculation of Wronskian (check!) or from the fact that the complex valued solutions $e^{\lambda t}$ and $e^{\bar{\lambda} t}$ form a fundamental set of solutions and relations (12)

16. Solve the following two differential equations which are important in applied mathematics:

$$y'' + \omega^2 y = 0 \quad \text{and} \quad y'' - \omega^2 y = 0,$$

where ω is a real positive constant.

17. **Alternative form of solution (13)**:

$$y(t) = e^{\alpha t} R\cos(\omega t - \delta), \quad (14)$$

where

$$R = \sqrt{C_1^2 + C_2^2}, \quad \cos\delta = \frac{C_1}{\sqrt{C_1^2 + C_2^2}} = \frac{C_1}{R}, \quad \sin\delta = \frac{C_2}{\sqrt{C_1^2 + C_2^2}} = \frac{C_2}{R}.$$

Note that $\tan\delta = C_2/C_1$.

18. Application: Mechanical unforced vibration: a mass hanging from a spring (more details in Section 3.7 that will be discussed later).
    - $\lambda = 0$ corresponds to **undamped** free vibration (simple harmonic motion)
    - $\lambda < 0$ corresponds to **damped** free vibration
    - R is called the **amplitude** of the motion
    - δ is called the **phase**, or phase angle, and measures the displacement of the wave from its normal position corresponding to $\delta = 0$.
    - $T = \dfrac{2\pi}{\omega}$ is the **quasi-period** of the motion.
19. Consider
$$y'' + 2y' + 3y = 0. \tag{15}$$
    (a) Find general solution.

    (b) Find solution of (15) subject to the initial conditions
$$y(0) = 2, \quad y'(0) = 1.$$
    (c) Determine α, $\omega > 0$, $R > 0$ and $\delta \in [0, 2\pi)$ so that the solution obtained in the previous item can be written in the form $e^{\alpha t} R\cos(\omega t - \delta)$, sketch the graph of the solution, and describe the behavior of the solution as t increases.

Linear homogeneous systems with constant coefficients: the case of complex eigenvalues

20. Let A be an $n \times n$ matrix with real entries. It may happen that the characteristic equation $\det(A - \lambda I) = 0$ has a complex root λ, i.e. λ is a complex eigenvalue of A. Then a correcponding eigenvector v has complex components, because it satisfy the linear algebraic system of equations $(A - \lambda I)v = 0$ and the matrix $A - \lambda I$ already have complex entries on its diagonal.

21. If λ is a complex eigenvalue of the real matrix A with an eigenvector v, then $\bar{\lambda}$ (the complex conjugate of λ) is an eigenvalue of A with an eigenvector $\bar{v}$. In other words complex eigenvalues of a real matrix A come in pair of complex conjugate ones.

22. Hence if λ is a complex eigenvalue of a real matrix A with an eigenvector v then both $e^{\lambda t}v$ and $e^{\bar{\lambda}t}\bar{v}$ are solutions of the system $X' = AX$. These solutions are vector values functions have complex components. Then superposition principle, the vector valued functions $\mathrm{Re}(e^{\lambda t}v)$ and $\mathrm{Im}(e^{\lambda t}v)$, i.e. the vector valued function of the real parts of the components of $e^{\lambda t}v$ and the vector valued function of the imaginary parts of the components of $e^{\lambda t}v$ are solution of the same system $X' = AX$:

23. Note that any vector v with complex component can be uniquely represented as $v = a + ib$ where a and b are vectors with real components, the real and imaginary parts of v respectively.

24. If $\lambda = \alpha + i\omega$, $\beta \neq 0$ is an eigenvalue of a real matrix A and $v = \mathbf{a} + i\mathbf{b}$ is an eigenvector of λ, where **a** and **b** are vectors with real components then

    $e^{t\lambda}v =$ i.e.

$$\boxed{\begin{aligned}\mathrm{Re}(e^{\lambda t}v) &= e^{\alpha t}(\cos(\omega t)\,\mathbf{a} - \sin(\omega t)\,\mathbf{b}),\\ \mathrm{Im}(e^{\lambda t}v) &= e^{\alpha t}(\sin(\omega t)\,\mathbf{a} + \cos(\omega t)\,\mathbf{b})\end{aligned}}$$

25. Case $n = 2$. If λ is a complex eigenvalue of the coefficient matrix A in the homogeneous system $X' = AX$ and $\mathbf{v}$ is a corresponding eigenvector then

    - $$\left\{e^{\lambda t}\mathbf{v}, e^{\bar{\lambda} t}\bar{\mathbf{v}}\right\} \tag{16}$$

      is a fundamental set of (complex) solutions of the system $X' = AX$.

    - $$\left\{\mathrm{Re}(e^{\lambda t}\mathbf{v}), \mathrm{Im}(e^{\lambda t}\mathbf{v})\right\} \tag{17}$$

      is a **real** fundamental set of solutions of the system $X' = AX$.

    We are interested in real solutions, so when you are asked for the general solutions then it means real general solutions and you have to convert (16) **into** (17) **and take the linear combination of what you obtained**.

26. Example. Consider $\begin{pmatrix} 3 & 1 \\ -5 & 1 \end{pmatrix}$

    (a) Find general solution of the system $X' = AX$.

    (b) Find solution subject to the initial conditions $x_1(0) = 2, x_2(0) = 3$.

27. Case $n = 3$. If $\alpha \pm i\omega$ are complex eigenvalue of the coefficient matrix A, then the third eigenvalue must be real (denote it by λ). Let **v** and **w** be eigenvectors corresponding to $\alpha + i\beta$ and λ, respectively. Then

$$\left\{\mathrm{Re}(e^{\lambda t}\mathbf{v}), \mathrm{Im}(e^{\lambda t}\mathbf{v}), e^{\lambda t}\mathbf{w}\right\}$$

    is a **real** fundamental set of solutions of the system $X' = AX$.