

## 16: Mechanical and Electrical Vibrations (section 3.7)

Consider linear dynamical system in which mathematical model is the following IVP:

$$au'' + bu' + cu = g(t), \quad u(0) = u_0, \quad u'(0) = v_0.$$

Here  $g(t)$  is **forcing function** of the system. Here we only discuss the case  $g(t) = 0$ , i.e. *free vibrations*, no external force. A solution  $u(t)$  of the ODE on an interval containing  $t = 0$  that satisfies the initial conditions is called the **response** of the system.

### Spring/mass systems: Free Undamped Vibration (or simple harmonic motion)

1. Spring on a table (horizontal spring.)

$$mu'' = -ku$$

2. A flexible spring is suspended vertically from a rigid support and the mass  $m$  is attached to the end. By **Hooke's Law**, the spring itself exerts a *restoring force*  $F$  opposite to the direction of elongation and proportional to the amount of elongation  $L$ :  $F = -kL$ , where  $k$  is called the **spring constant**.

3. The mass  $m$  stretches the spring by  $L$  and attains a position of equilibrium, i.e. weight,  $mg$ , is balanced by the restoring force:

$$mg - kL = 0.$$

4. If the mass is displaced by an amount  $u$  from its equilibrium position, the restoring force is then  $-k(u + L)$ . **Free motion** (i.e. no other external/retarding forces acting on the moving mass): use **Newton's second Law** with the net (or resultant) force:

$$mu'' = -k(u + L) + mg = -ku.$$

5. ODE of Free Undamped Motion:

$$u'' + \omega_0^2 u = 0, \tag{1}$$

where

$$\omega_0^2 = \frac{k}{m}.$$

Initial conditions:  $u(0) = u_0$ ,  $u'(0) = v_0$ , where  $u_0$  is the initial displacement and  $v_0$  is the initial velocity. For example,  $u_0 < 0$  and  $v_0 = 0$  mean that the mass is released from rest from a point  $|u_0|$  units above the equilibrium position.

General solution of (1) is

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = R \cos(\omega_0 t - \delta),$$

where

- $R = \sqrt{C_1^2 + C_2^2}$  is called the **amplitude** of the motion

- $\delta$  is called the **phase**, or phase angle, and measures the displacement of the wave from its normal position corresponding to  $\delta = 0$ . Recall that

$$\cos \delta = \frac{C_1}{\sqrt{C_1^2 + C_2^2}} = \frac{C_1}{R}, \quad \sin \delta = \frac{C_2}{\sqrt{C_1^2 + C_2^2}} = \frac{C_2}{R}.$$

- $\omega_0 = \sqrt{\frac{k}{m}}$  is the **natural frequency** of the system.
  - $T = \frac{2\pi}{\omega_0}$  is the **period** of the motion. The number  $T$  is time it takes the mass to execute one cycle of motion (the length of the interval between two successive maxima (or minima) of  $u(t)$ .)
  - The **frequency of motion**  $f = \frac{1}{T} = \frac{\omega_0}{2\pi}$ .
6. A mass weighing 4lb stretches a spring 6 inches. At  $t = 0$  the mass released from a point 8 inches below the equilibrium with an upward velocity of  $\frac{2}{3}$  ft/s. Determine the amplitude of vibrations, phase angle, period, natural frequency of the system and frequency of motion.

### Spring/mass systems: Free Damped Vibrations.

7. Assume that the mass is suspended in a viscous medium or connected to a dashpot damping device. Dampers work to counteract any movement: damping force =  $-\gamma v = -\gamma u'$ , where  $\gamma$  is a positive damping constant.
8. ODE of Free Damped Motion:

$$mu'' + \gamma u' + ku = 0. \quad (2)$$

9. Discriminant of the characteristic equation  $m\lambda^2 + \gamma\lambda + k = 0$  is

$$D = \gamma^2 - 4mk.$$

10. CASE 1: (Underdamping)  $D < 0$ , i.e. the roots are complex conjugate:

$$\lambda_{1,2} = -\frac{\gamma}{2m} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}},$$

where  $\omega_0^2 = \frac{k}{m}$ .

General solution of (2) is not periodic:

$$u(t) = e^{-\frac{\gamma}{2m}t} \left( \cos(\omega t) + C_2 \sin(\omega t) \right) = Re^{-\frac{\gamma}{2m}t} \cos(\omega t - \delta),$$

where

- $Re^{-\frac{\gamma}{2m}t}$  is **damped amplitude** of vibrations.

- $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}}$  is the **quasi frequency**; here  $\omega_0^2 = \frac{k}{m}$ ,  $\omega_0$  is the **natural frequency** of the system (i.e. without damping). Note that  $\omega < \omega_0$ , if there is a damping.
- $T_d = \frac{2\pi}{\omega}$  is the **quasi period**, i.e. the time interval between two successive maxima of  $u(t)$ .

Note that as the damping coefficient  $\gamma$  increases, the quasi frequency  $\mu$  becomes smaller and the quasi period becomes bigger.

11. CASE 2: (Critical Damping)  $D = 0$  (two repeated (equal) roots) or equivalently

$$\boxed{\gamma_{\text{crit}} = 2\sqrt{mk}}.$$

The motivation for the term “critical”:

- If  $0 < \gamma < \gamma_{\text{crit}}$ , then  $D > 0$  the motion is **oscillatory** in the following sense: the mass passes the equilibrium infinitely many times, although the amplitude of oscillation decays exponentially;
- (Overdamping) If  $\gamma > \gamma_{\text{crit}}$ , then  $D > 0$  (two distinct real roots  $\lambda_1$  and  $\lambda_2$ ). In this case there are no oscillation. The general solution of (2) has no more than one zero:

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

so the motion is **non-oscillatory**.

**Conclusion:** *The critical damping is the transitional value of the damping from oscillatory to non-oscillatory motion.*

For  $\gamma = \gamma_{\text{crit}}$  the characteristic equation has repeated roots. The general solution of equation of type (2) will be discussed in the next section.

### LRC electrical circuit

12. If  $Q$  is the charge at time  $t$  in an electrical closed circuit with inductance  $L$ , resistance  $R$ , and capacitance  $C$ , then by Kirchoff’s Second Law (from Physics) the impressed voltage  $E(t)$  is equal to the sum of the voltage drops in the rest of the circuit

$$E(t) = IR + \frac{Q}{C} + LI'(t).$$

By substitution  $I = Q'$  we get

$$LQ'' + RQ' + \frac{1}{C}Q = E(t).$$

Analogy between electrical and mechanical quantities:

Charge $Q$	Position $u$
Inductance $L$	mass $m$
Resistance $R$	Damping constant $\gamma$
Inverse capacitance $1/C$	Spring constant $k$
Impressed voltage $E(t)$ (electromotive force)	External force $F(t)$