16: Mechanical and Electrical Vibrations (section 3.7)

Consider linear dynamical system in which mathematical model is the following IVP:

$$au'' + bu' + cu = g(t), \quad u(0) = u_0, \quad u'(0) = v_0.$$

Here $g(t)$ is **forcing function** of the system. Here we only discuss the case $g(t) = 0$, i.e *free vibrations*, no external force. A solution $u(t)$ of the ODE on an interval containing $t = 0$ that satisfies the initial conditions is called the **response** of the system.

Spring/mass systems: Free Undamped Vibration (or simple harmonic motion)

1. Spring on a table (horizontal spring).)

$$mu'' = -ku$$

2. A flexible spring is suspended vertically from a rigid support and the mass m is attached to the end. By **Hooke's Law**, the spring itself exerts a *restoring force* F opposite to the direction of elongation and proportional to the amount of elongation L: $F = -kL$, where k is called the **spring constant**.

3. The mass m stretches the spring by L and attains a position of equilibrium, i.e. weight, mg, is balanced by the restoring force:
$$mg - kL = 0.$$

4. If the mass is displaced by an amount u from its equilibrium position, the restoring force is then $-k(u + L)$. Free motion (i.e. no other external/retarding forces acting on the moving

mass): use Newton's second Law with the net (or resultant) force:

$$mu'' = -k(u+L) + mg = -ku.$$

5. ODE of Free Undamped Motion:

$$u'' + \omega_0^2 u = 0, \tag{1}$$

where

$$\omega_0^2 = \frac{k}{m}.$$

Initial conditions: $u(0) = u_0, \quad u'(0) = v_0$, where u_0 is the initial displacement and v_0 is the initial velocity. For example, $u_0 < 0$ and $v_0 = 0$ mean that the mass is released from rest from a point $|u_0|$ units above the equilibrium position.

General solution of (1) is

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = R\cos(\omega_0 t - \delta),$$

where

- $R = \sqrt{C_1^2 + C_2^2}$ is called the **amplitude** of the motion
- δ is called the **phase**, or phase angle, and measures the displacement of the wave from its normal position corresponding to $\delta = 0$. Recall that

$$\cos\delta = \frac{C_1}{\sqrt{C_1^2 + C_2^2}} = \frac{C_1}{R}, \quad \sin\delta = \frac{C_2}{\sqrt{C_1^2 + C_2^2}} = \frac{C_2}{R}.$$

- $\omega_0 = \sqrt{\dfrac{k}{m}}$ is the **natural frequency** of the system.
- $T = \dfrac{2\pi}{\omega_0}$ is the **period** of the motion. The number T is time it takes the mass to execute one cycle of motion (the length of the interval between two successive maxima (or minima) of $u(t)$.)

- The **frequency of motion** $f = \frac{1}{T} = \frac{\omega_0}{2\pi}$.

6. A mass weighing 4lb stretches a spring 6 inches. At $t = 0$ the mass released from a point 8 inches below the equilibrium with an upward velocity of $\frac{2}{3}$ ft/s. Determine the amplitude of vibrations, phase angle, period, natural frequency of the system and frequency of motion.

Spring/mass systems: Free Damped Vibrations.

7. Assume that the mass is suspended in a viscous medium or connected to a dashpot damping device. Dampers work to counteract any movement: damping force $= -\gamma v = -\gamma u'$, where γ is a positive damping constant.

8. ODE of Free Damped Motion:
$$mu'' + \gamma u' + ku = 0. \tag{2}$$

9. Discriminant of the characteristic equation $m\lambda^2 + \gamma\lambda + k = 0$ is
$$D = \gamma^2 - 4mk.$$

10. CASE 1: (Underdamping) $D < 0$, i.e. the roots are complex conjugate:
$$\lambda_{1,2} = -\frac{\gamma}{2m} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}},$$
where $\omega_0^2 = \dfrac{k}{m}$.

General solution of (2) is not periodic:
$$u(t) = e^{-\frac{\gamma}{2m}t}\Big(\cos(\omega t) + C_2\sin(\omega t)\Big) = Re^{-\frac{\gamma}{2m}t}\cos(\omega t - \delta),$$
where

- $Re^{-\frac{\gamma}{2m}t}$ is **damped amplitude** of vibrations.
- $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}}$ is the **quasi frequency**; here $\omega_0^2 = \dfrac{k}{m}$, ω_0 is the **natural frequency** of the system (i.e. without damping). Note that $\omega < \omega_0$, if there is a damping.

- $T_d = \frac{2\pi}{\omega}$ is the **quasi period**, i.e. the time interval between two successive maxima of $u(t)$.

Note that as the damping coefficient γ increases, the quasi frequency μ becomes smaller and the quasi period becomes bigger.

11. CASE 2: (Critical Damping) $D = 0$ (two repeated (equal) roots) or equivalently

$$\boxed{\gamma_{\text{crit}} = 2\sqrt{mk}}.$$

The motivation for the term "critical":

(a) If $0 < \gamma < \gamma_{\text{crit}}$, then $D > 0$ the motion is **oscillatory** in the following sense: the mass passes the equilibrium infinitely many times, although the amplitude of oscillation decays exponentially;

(b) (Overdamping) If $\gamma > \gamma_{\text{crit}}$, then $D > 0$ (two distinct real roots λ_1 and λ_2).In this case there are no oscillation. The general solution of (2) has no more than one zero:

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

so the motion is **non-oscillatory**.

Conclusion: *The critical dumping is the transitional value of the damping from oscillatory to non-oscillatory motion.*

For $\gamma = \gamma_{\text{crit}}$ the characteristic equation has repeated roots. The general solution of equation of type (2) will be discussed in the next section.

LRC electrical circuit

12. If Q is the charge at time t in an electrical closed circuit with inductance L, resistance R, and capacitance C, then by Kirchhoff's Second Law (from Physics) the impressed voltage $E(t)$ is equal to the sum of the voltage drops in the rest of the circuit

$$E(t) = IR + \frac{Q}{C} + LI'(t).$$

By substitution $I = Q'$ we get

$$LQ'' + RQ' + \frac{1}{C}Q = E(t).$$

Analogy between electrical and mechanical quantities:

Charge Q	Position u
Inductance L	mass m
Resistance R	Damping constant γ
Inverse capacitance $1/C$	Spring constant k
Impressed voltage $E(t)$ (electromotive force)	External force $F(t)$