17. The case of equal (or repeated) roots for second order homogeneous equations (section 3.4) and toward the case of repeated eigenvalues for systems: matrix exponential (section 7.7)

The case of equal (or repeated) roots for second order homogeneous equations (section 3.4)

1. Recall that the **characteristic equation** of a linear homogeneous equation with constant real coefficients
$$ay'' + by' + cy = 0 \tag{1}$$
is
$$a\lambda^2 + b\lambda + c = 0. \tag{2}$$
Assume that
$$D = b^2 - 4ac = 0 \Rightarrow \lambda_1 = \lambda_2 = -\frac{b}{2a} := \lambda$$
So, we know how to one particular solution $y_1(t) = e^{\lambda t}$.

2. How to choose a second particular solution y_2 such that the set $\{y_1, y_2\}$ will be fundamental, i.e. $W(y_1, y_2) \neq 0$?:
The answer: $y_2(t) = te^{\lambda t}$ so that the general solution is
$$\boxed{y(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}} = (C_1 + C_2 t)e^{\lambda t}.$$

3. Here is a "Physics style" explanation, which considers **the case of repeated roots as a limiting case of the case of distinct real roots** (I will post two other explanations, one based on factorization of second order differential operator and another based on the method of reduction of order, in the Enrichment)
EXPLANATION

4. Find the general solution of $y'' - 6y' + 9y = 0$.

SUMMARY:

Solution of linear homogeneous equation of second order with constant coefficients
$$ay'' + by' + cy = 0$$

Sign of $D = b^2 - 4ac$	Roots of characteristic polynomial $a\lambda^2 + b\lambda + c = 0$	General solution
$D > 0$	two distinct real roots $\lambda_1 \neq \lambda_2$	$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
$D < 0$	two complex conjugate roots $\lambda_1 = \overline{\lambda_2}$: $\lambda_{1,2} = \alpha \pm i\omega$	$y(t) = C_1 e^{\alpha t}\cos(\omega t) + C_2 e^{\alpha t}\sin(\omega t)$
$D = 0$	two equal(repeated) real roots $\lambda_1 = \lambda_2 = \lambda$	$y(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$

Toward the case of repeated eigenvalues: matrix exponential (section 7.7)

5. Recall that the Taylor expansion of the exponential function e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!} \tag{3}$$

which implies that

$$e^{\lambda t} = 1 + t\lambda + \frac{t^2}{2!}\lambda^2 + \frac{t^3}{3!}\lambda^3 + \ldots = \sum_{i=0}^{\infty} \frac{t^i}{i!}\lambda^i. \tag{4}$$

In particular, in the previous notes, section 15, item 11 there, we used this formula to prove that

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t} \tag{5}$$

for any complex λ.

6. Now replace the number x by an $n \times n$ matrix A in (3) (replacing also the number 1 in the very first term by the $n \times n$ identity matrix I) to obtain another $n \times n$ matrix e^A called the *matrix exponential of* A:

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots = \sum_{i=0}^{\infty} \frac{1}{i!}A^i \tag{6}$$

Then we can consider e^{tA} (a matrix-values function assigning to any t the matrix exponential of the matrix tA),

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots = \sum_{i=0}^{\infty} \frac{t^i}{i!}A^i. \tag{7}$$

and exactly as for (8) one can obtain from this that

$$\frac{d}{dt}e^{tA} = Ae^{tA} \tag{8}$$

This implies that *for any column vector v the vector function $e^{tA}v$ is a solution of the system $X' = AX$.*

7. One says that n vectors $v^1, \ldots, v^n$ constitute a basis in $\mathbb{R}^n$ if any other vector v in $\mathbb{R}^n$ can be uniquely represented as a linear combination of $v^1, \ldots, v^n$, i.e. there exist constants $c_1, \ldots, c_n$ such that

$$v = c_1 v^1 + \ldots c_n v^n.$$

Equivalently, $v^1, \ldots, v^n$ constitute a basis in $\mathbb{R}^n$ if and only if $\det(v^1, \ldots, v^n) \neq 0$.

Based on the previous item, if $v^1, \ldots v^n$ constitute a basis of $\mathbb{R}^n$, then

$$\{e^{tA}v^1, e^{tA}v^2, \ldots, e^{tA}v^n\} \tag{9}$$

form a fundamental set of solutions of A

The big question: How to calculate e^{At} or how to find a convenient basis $v^1, \ldots, v^n$ for which $e^{At}v^i$ can be effectively calculated for every $i = 1, \ldots, n$?

8. Note that in contrast to numbers for matrices $e^A e^B \neq e^B e^A$. However, if the matrices A and B commute, $AB = BA$ then $e^A e^B = e^B e^A$ (just take the products of their Taylor expansion and use commutativity). Moreover, if $AB = BA$, then $e^{A+B} = e^A e^B$, which is not true in general.

9. If $A = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$, the diagonal matrix with entries $\lambda_1, \dots, \lambda_n$ on the diagonal, then $e^{tA} = \operatorname{diag}\{e^{t\lambda_1}, \dots, e^{t\lambda_n}\}$

   In particular, if $A = \lambda I$, A is the diagonal matrix with all entries on the diagonal being equal, then $e^{t\lambda I} = e^{\lambda t} I$

10. Note that the matrix λI commutes with any other matrix. Therefore using the previous two items we can get the following formula that will be crucial in the sequal:

$$\boxed{e^{tA} = e^{\lambda t} e^{t(A-\lambda I)}} \tag{10}$$

11. Why the formula (10) is useful. Assume that λ is an eigenvalue and v is the corresponding eigenvector. Then calculate $e^{tA}v$ using (10)

    So, $e^{tA}v = e^{\lambda t}v$ as expected by our previous considerations (see section 12, item 4.)

12. Conclusion: As a consequence of item 3 above (see the sentence including formula (9) , if an $n \times n$ matrix A admits a basis of eigenvectors $v^1, \dots v^n$ in $\mathbb{R}^n$, then $(e^{t\lambda_1}v^1, \dots, e^{t\lambda_n}v^n)$ form a fundamental set of solutions of $X' = AX$.

    Note that we could derive it without using matrix exponential and (9), but the matrix exponential gives a way that works in more general situation.

13. **The life is not so simple: not any $n \times n$ matrix admits a basis of eigenvectors in $\mathbb{R}^n$.**

    EXAMPLE 1. *Let* $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

    (a) *Find all eigenvectors of N. Can you choose eigenvectors of N that constitute a basis of* $\mathbb{R}^2$

    (b) *Calculate* e^{tN}.