18: Repeated Eigenvalues: algebraic and geometric multiplicities of eigenvalues, generalized eigenvectors, and solution for systems of differential equation with repeated eigenvalues in case $n = 2$ (sec. 7.8)

1. We have seen that not every matrix admits a basis of eigenvectors. First, discuss a way how to determine if there is such basis or not.

   Recall the following two equivalent characterization of an eigenvalue:

   (a) λ is an eigenvalue of $A \Leftrightarrow \det(A - \lambda I) = 0$;

   (b) λ is an eigenvalue of $A \Leftrightarrow$ there exist a nonzero vector v such that $(A - \lambda I)v = 0$. The set of all such vectors together with the 0 vector form a vector space called the *eigenspace of* λ and denoted by E_λ.

   Based on these two characterizations of an eigenvalue λ of a matrix A one can assign to λ the following two positive integer numbers,

   - **Algebraic multiplicity** of λ is the multiplicity of λ in the characteristic polynomial $\det(A - xI)$, i.e. the maximal number of appearances of the factor $(x - \lambda)$ in the factorization of the polynomial $\det(A - xI)$.
   - **Geometric multiplicity** of λ is the dimension $\dim E_\lambda$ of the eigenspace of λ, i.e. the maximal number of linearly independent eigenvectors of λ.

2. For example, if $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (as in the example in item 13 of the previous notes), then $\lambda = 0$ is the unique eigenvalue. Find the algebraic multiplicity and geometric multiplicity of $\lambda = 0$

THEOREM 1. *Geometric multiplicity is not greater than algebraic multiplicity.*

THEOREM 2. *A matrix A admits a basis of eigenvectors if and only of for every its eigenvalue λ the geometric multiplicity of λ is equal to the algebraic multiplicity of λ.*

REMARK 3. *In Linear Algebra matrices admitting a basis of eigenvectors are called* diagonizable *(because they are diagonal in this basis).*

REMARK 4. *Basis of eigenvectors always exists for the following classes of matrix:*

- **symmetric matrices***: $A^T = A$, or equivalently, $a_{ij} = a_{ji}$ for all i, j;*
- **skew-symmetric** *$A^T = -A$, or equivalently, $a_{ij} = -a_{ji}$ for all i, j.*

For symmetric matrices all eigenvalues are real and the the eigenspaces corresponding to the different eigenvalues are orthogonal. For skew-symmetric matrices the eigenvalues are purely imaginary (i.e. of the form $i\beta$).

3. If the matrix A does not admit a basis of eigenvectors then for what vectors w other than the eigenvectors it is still easy to calculate $e^{At}w$ in the light of the formula

$$\boxed{e^{tA} = e^{\lambda t}e^{t(A-\lambda I)}} \tag{1}$$

(see item 6 of the previous lecture notes)?

4. Assume that w is such that

$$(A - \lambda I)w \neq 0, \text{ but } (A - \lambda I)^2 w = 0 \tag{2}$$

(the first relation means that w is not an eigenvector corresponding to λ). Calculate $e^{At}w$ using (1).

5. More generally if we assume that for some $k > 0$

$$(A - \lambda I)^{k-1} w \neq 0, \text{ but } (A - \lambda I)^k w = 0 \tag{3}$$

then $e^{At}w$ can be calculated using only finite number of terms when expanding $e^{t(A-\lambda I)}w$ from (1).

Note that if for some λ there exists w satisfying (3) then λ must be an eigenvalue

6.

DEFINITION 5. *A vector w satisfying* (3) *for some $k > 0$ is called a* generalized eigenvector *of λ (of order k).*

The set of all generalized eigenvectors of λ together with the 0 vector is a vector space denoted by E_λ^{gen}

REMARK 6. *The (regular) eigenvector is a generalized eigenvector of order* 1*, so $E_\lambda \subset E_\lambda^{\text{gen}}$ (given two sets A and B, the notation $A \subset B$ means that the set A is a subset of the set B, i.e. any element of the set A belongs also to B)*

THEOREM 7. *The dimension of the space E_λ^{gen} of generalized eigenvectors of λ is equal to the algebraic multiplicity of λ.*

THEOREM 8. *Any matrix A admits a basis of generalized eigenvectors.*

Let us see how it all works in the first nontrivial case of $n = 2$.

7. Let A be 2×2 matrix and λ is a repeated eigenvalue of A. Then its algebraic multiplicity is equal to _

   There are two options for the geometric multiplicity:

   1 (trivial case) Geometric multiplicity of λ is equal to 2. Then $A = \lambda I$

   2. (less trivial case) Geometric multiplicity λ is equal to 1. In the rest of these notes we concentrate on this case only.

8.

   PROPOSITION 9. *Let w be a nonzero vector which is not an eigenvector of λ, $w \notin E_\lambda$. The vector w satisfies $(A-\lambda I)^2 w = 0$, i.e. w is a generalized eigenvector of order 2. Besides, in this case*

$$v := (A - \lambda I)w \tag{4}$$

   is an eigenvector of A corresponding to λ.

9. Note that $\{v, w\}$ constructed above constitute a basis of $\mathbb{R}^2$ (i.e. $E_\lambda^{\text{gen}} = \mathbb{R}^2$, so we proved Theorem 8 in this case. Therefore, $\{e^{tA}v, e^{tA}w\}$ form a fundamental set of solutions for the system $X' = AX$. By constructions and calculation as in item 4 above

   $e^{tA}v =$

   $e^{tA}w =$

   Conclusion:

$$\boxed{\{e^{\lambda t}v, e^{\lambda t}(w + tv)\}}. \tag{5}$$

   form a fundamental set of solutions of $X' = AX$, i.e. the general solution is

$$\boxed{e^{\lambda t}(C_1 v + C_2(w + tv))}. \tag{6}$$

10. This gives us the following algorithms for fining the fundamental set of solutions in the case of a repeated eigenvalue λ with geometric multiplicity 1.

    Algorithm 1 (easier than the one in the book):

    (a) Find the eigenspace E_λ of λ by finding all solutions of the system $(A - \lambda I)v = 0$. The dimension of this eigenspace under our assumptions must be equal to _.

    (b) Take any vector w not lying in the eigenline E_λ and find $v := (A - \lambda I)w$. With chosen v and w the general solution is given by (6).

    Algorithm 2 (as in the book):

    (a) Find an eigenvector v by finding one nonzero solution of the system $(A - \lambda I)v = 0$.

    (b) With v found in item 1 find w suh that $(A - \lambda I)w = v$. With chosen v and w the general solution is given by (6).

    REMARK 10. *The advantage of Algorithm 1 over Algorithm 2 is that in the first one you solve only one linear system when finding the eigenline, while in Algorithm 2 you need to solve one more linear system $(A - \lambda I)w = v$ for w (in Algorithm 1 you choose w and then find v from* (4) *instead).*

11. Finally let us give another algorithm which works only in the case $n = 2$ (for higher n it works only under some additional assumption that A has only one eigenvalue). This algorithm does not use eigenvetors explicitly (although implicitly we use here the information that an eigenvalue λ is repeated). Proposition 9 actually implies that

$$(A - \lambda I)^2 = 0. \tag{7}$$

then based on (1) and (7)

$e^{tA} =$

Conclusion:

$$e^{At} = e^{\lambda t}(I + t(A - \lambda I)) \tag{8}$$

Algorithm 3: Calculate e^{tA} from (8). The columns of the resulting matrix form a fundamental set of solutions.

REMARK 11. *Identity* (7) *is in fact a particular case of the followiing remarkable result from Linear Algebra, called Caley-Hamilton: Let*

$$\det(A - \lambda I) = (-1)^n(\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0) \tag{9}$$

then

$$A^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I = 0 \tag{10}$$

In other words, if one substitutes the matrix A instead of λ and $a_0 I$ instead of a_0 into the right hand side of (10) *then you will get* 0.

12. Example. Find general solution of the system.: $\begin{cases} x_1' &= -3x_1 + \frac{5}{2}x_2 \\ x_2' &= -\frac{5}{2}x_1 + 2x_2 \end{cases}$

13. Now return to a second order linear homogeneous equation

$$ay'' + by' + cy = 0 \tag{11}$$

with a repeated root of its characteristic polynomial. How to explain that $\{e^{\lambda t}, te^{\lambda t}\}$ is a fundamental set of solution from the theory of systems of first equations with repeated eigenvalues?

Consider the corresponding system

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{c}{a}x_1 - \frac{b}{a}x_2. \end{cases} \tag{12}$$

The point here that one can take

$$v = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so the first component of $e^{\lambda t}v$ is equal to $e^{\lambda t}$ and the first component of $e^{\lambda t}(w + tv)$ is equal to $te^{\lambda t}$.