20: Non-homogeneous Linear Systems : method of variation of parameters (section 7.9) with application to non-homogeneous linear equations of second order (section 3.6)

1. Consider a non-homogeneous Linear system

$$X' = P(t)X + G(t), \tag{1}$$

where

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \dots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \dots & p_{2n}(t) \\ \vdots & & & \vdots \\ p_{n1}(t) & p_{n2}(t) & \dots & p_{nn}(t) \end{pmatrix}, \quad G(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix},$$

The system

$$X' = P(t)X. \tag{2}$$

is called the *homogeneous system associated with the nonhomogeneous system* (1).

2. Relation between solutions of (1) and (2) can be formulated in the following:

PROPOSITION 1.

(a) *If $X^1(t)$ and $X^2(t)$ are two solutions of the nonhomogeneous system* (1) *then their difference $X^1(t) - X^2(t)$ is a solution of the associated homogeneous system* (2)

(b) *If $X_p(t)$ is a solution of the nonhomogeneous system* (1) *(the subscript p stands for "particular") and $X_h(t)$ is a solution of the corresponding homogeneous equation* (2), *then their sum $X_p(t) + X_h(t)$ is a solution of the nonhomogeneous system* (1)

Proof

COROLLARY 2. *The general solution of the nonhomogeneous system* (1) *is the sum of a particular (i.e. specific) solution of it and the general solution of the corresponding homogeneous system* (2).

Method of Variation of Parameters

3. The method of variation of parameter gives an algorithm to find a particular (and therefore the general) solution of the nonhomogeneous system from the knowledge of the general solution of the corresponding homogeneous system (2):

   Suppose that $\{X^1(t), \dots, X^n(t)\}$ is a fundamental set of solutions of the corresponding homogeneous system (2). Then the general solution of the homogeneous system (2) is

$$X(t) = C_1 X^1(t) + \dots C_n X^n(t), \tag{3}$$

   where $C_1, \dots, C_n$ are arbitrary constants.

   The main idea of the method of *variation of parameters* (called also *variation of constants*) is to look for a particular solution of the nonhomogeneous equation (1) in the form

$$X(t) = u_1(t) X^1(t) + \dots u_n(t) X^n(t) \tag{4}$$

   for some unknown functions $u_1(t), \dots, u_n(t)$. Note that (4) is obtained from (3) by replacing constants $C_1, \dots, C_n$ by functions $u_1(t), \dots, u_n(t)$ and this is the reason for the name of the method (we variate constants (parameters) C_i by functions u_i). Substituting expression (1) into the original nonhomogeneous system (1) one gets a linear algebraic system of equation for derivatives $u_1'(t), \dots, u_n'(t)$.

4. To deduce this system of linear algebraic equations for $u_1'(t), \dots, u_n'(t)$ it is more convenient to write (4) in the matrix form. Consider the so called **fundamental matrix**, $\Psi(t)$, of the homogeneous system (2) whose columns are vectors $X^1(t), \dots, X^n(t)$ [1]

$$\Psi(t) = \begin{pmatrix} x_{11}(t) & \dots & x_{1n}(t) \\ \vdots & \vdots & \vdots \\ x_{n1}(t) & \dots & x_{nn}(t) \end{pmatrix}.$$

Then

(a) Since each columns of the matrix $\Psi(t)$ is a solution of (2), the matrix $\Psi(t)$ satisfies the following matrix differential equation

$$\Psi'(t) = P(t)\Psi(t) \tag{5}$$

(b) Equations (4) can be written as

$$X(t) = \Psi(t)U(t), \tag{6}$$

where $U(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}$.

5. Substitute (6) into (1) and use (5):

Conclusion: You get the following linear algebraic system of equation for $U'(t)$

$$\Psi(t)U'(t) = G(t). \tag{7}$$

[1] Recall that $\det \Psi(t) = W[X^1, \dots, X^n](t)$

6. Then you proceed as follows:

   (a) Find $U'(t)$ from (7). You can use any method for solving linear algebraic systems you know: the Gauss elimination, substitution, the Cramer rule, inverse matrices (the notion of the inverse matrix is discussed in Appendix at the end of these notes)

   (b) Integrate each component of a vector function found in the previous item to get $U(t)$;

   (c) Substitute $U(t)$ into (6) or, equivalently, into (4) to get a particular solution of (1);

   (d) Use Corollary 2 and equation (3) to get the general solution.

   REMARK 3. *One can obtain item (d) immediately from item (c) if one does not specify constants when integrating in item (b).*

   REMARK 4. *Using inverse matrix one can write the general solution in one line formula*

$$X(t) = \Psi(t)C + \int_0^t \Psi(t)\Psi^{-1}(\tau)g(\tau)\mathrm{d}\tau. \tag{8}$$

   In particular if $P(t) \equiv A$, a constant matrix, i.e. (2) has the form $X'(t) = AX$ then e^{tA} is the fundamental matrix and (4) can be written as

$$X(t) = e^{tA}X(0) + \int_0^t e^{(t-\tau)A}g(\tau)\mathrm{d}\tau. \tag{9}$$

   (here we used that $\left(e^{tA}\right)^{-1} = e^{-tA}$ and that $e^{tA}e^{-\tau A} = e^{(t-\tau)A}$. We also use that $e^0 = I$ to get that $C = X(0)$). Formulas (8) and its particular case (9) are very useful to study general properties of linear nonhomogeneous system, for example they are basic tools in Theory of Linear Control Systems. However, to apply the method you do not need to memorize these formula or to use inverse matrices, just you need to remember the general scheme described in steps (a)-(d) above.

7. Example Find the general solution of the system:

$$\begin{cases} x_1' = -2x_1 + x_2 + e^{-t} \\ x_2' = x_1 - 2x_2 - e^{-t} \end{cases}$$

Applications to second order nonhomogeneous linear ODEs (section 3.6)

8. Consider a second-order *nonhomogeneous* linear DE

$$y'' + p(t)y' + q(t)y = g(t). \tag{10}$$

and associate homogeneous DE

$$y'' + p(t)y' + q(t)y = 0. \tag{11}$$

9. We know that (10) is equivalent to the following nonhomogeneous linear system of two first order equations:

$$\begin{cases} x_1' = x_2 \\ x_2' = -q(t)x_1 - p(t)x_2 + g(t) \end{cases} \tag{12}$$

or to system (1) with $n = 2$,

$$P(t) = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix}, \quad G(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

10. Apply the method of variation of parameter to this system:

If $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions of the homogeneous system (11), then

$$X^1(t) = \begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix}, \quad X^1(t) = \begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix}$$

is a fundamental set of solutions of the homogeneous system associated with the nonhomogeneous system (10), so the matrix

$$\Psi(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$$

is the fundamental matrix of this homogeneous system.

11. The application of the method of variation of parameter to the system (12) can be implemented as follows:

   (a) Find a solution of the original nonhomogeneous second order equation (10) in the form

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \tag{13}$$

which is equivalent to finding the solution of the system (12) in the form

$$u_1(t)\begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix} + u_2(t)\begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix}.$$

(b) The algebraic system (7) in this case is nothing but

$$\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \tag{14}$$

Find $u_1'(t)$ and $u_2'(t)$ from this system

(c) Proceed as in the steps (b)-(d) of the general procedure described in item 6 above.

12. Example Find the general solution of the equation:

$$y'' + y = \tan t, \quad 0 < t < \frac{\pi}{2}.$$

vfill

Appendix: Matrix Inverse (very briefly)

13. Let A be a square matrix of size n. A square matrix, A^{-1}, of size n, such that $AA^{-1} = I_n$ (or , equivalently, $A^{-1}A = I_n$) is called an **inverse matrix**.

14. A^{-1} in the case $n = 2$: If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

15. *FACT (Equivalent to Theorem 1 of section 11*: A^{-1} exists if and only if $\det A \neq 0$.

16. **Solving Systems of Equations with Inverses.**

    Let $AX = B$ be a linear system of n equations in n unknowns and A^{-1} exists, then $X = A^{-1}B$ is the *unique* solution of the system.