# 21: Nonhomogeneous Equations. Method of Undetermined Coefficients(judicious guess) (section 3.5) with applications to the forced vibrations (section 3.8)

## Method of Undetermined Coefficients

1. Consider a particular class of nonhomogeneous linear ODE with constant coefficients
$$ay'' + by' + cy = g(t),$$
where $a, b, c$ are real constants and $g(t)$ involves linear combinations, sums and products of
$$t^m, \quad e^{\alpha t}, \quad \sin(\beta t), \quad \cos(\beta t).$$

### The case when $g(t) = e^{\alpha t} P_n(t)$, where $P_n(t)$ is a polynomial of degree $n$

2. Consider the equation
$$ay'' + by' + cy = e^{\alpha t} P_n(t) \tag{1}$$

3. Since derivatives of any order of the product of $e^{\alpha t}$ and a polynomial is again the product of $e^{\alpha t}$ and a polynomial, it is natural to look for a particular solution of (1) $e^{\alpha t}$ and an unknown (undetermined) polynomial $Q(t)$. Plugging such function into equation (1) and comparing the right-hand side with the left-hand side we we will get $n+1$ linear equations for the coefficients of polynomial $Q$ (the undetermined coefficients) ($n+1$ is because a polynomial $P_n$ has $n+1$ coefficients). So you may expect that if the undetermined polynomial $Q(t)$ will depend on $n+1$ undetermined coefficients we will be able to find a solution of this system.

4. However, the life is not so easy and the form of undetermined polynomial will depend on *whether $\alpha$ is a root of the characteristic polynomial of the homogeneous equation corresponding to* (1)*, and if it is a root it will depend on the multiplicity of $\alpha$ in this characteristic polynomial*. In more detail

   (a) **If $\alpha$ is NOT a root of characteristic polynomial**, then we look for a solution in the form
   $$y_p(t) = e^{\alpha t}(A_0 + A_1 t + A_2 t^2 + \ldots + A_n t^n) \tag{2}$$
   so we have $n+1$ undetermined coefficients $A_0, \ldots, A_n$ that we can determine by plugging $y_p(t)$ into (1) and comparing coefficients of the left and right-hand sides.

   (b) (heuristic explanation) **If $\alpha$ is a root of multiplicity** 1 [1] , then $e^{\alpha t}$ is a solution of

[1] everywhere here multiplicity means the algebraic multiplicity

the corresponding homogeneous equation, so if we plug $y_p(t)$ of the form (2) into (1) $A_0$ will actually not appear there.

So, the number of "essential" undetermined coefficients will be $n$, which will be not enough for obtaining the right-hand side of (1) (actually one can show that the left-hand side and right-hand sides are never equal after plugging $y_p(t)$ of the form (2) in this case (because the coefficient of $e^\alpha t^n$ of the left-hand side will be 0, while the analogous coefficient of the right-hand side is not).

However, everything will work if we look for the solution in the form

$$y_p(t) = te^{\alpha t}(A_0 + A_1 t + A_2 t^2 + \ldots + A_n t^n) \tag{3}$$

so we have $n+1$ undetermined coefficients $A_0, \ldots, A_n$ that we can determine by plugging $y_p(t)$ into (1) and comparing coefficients of the left and right-hand sides.

(c) Similarly, **if $\alpha$ is a root of multiplicity** 2, then we look for the solution in the form

$$y_p(t) = t^2 e^{\alpha t}(A_0 + A_1 t + A_2 t^2 + \ldots + A_n t^n) \tag{4}$$

5. All three case can be unified if we will say that for $\alpha$ not being a root of the characteristic polynomial its multiplicity in this polynomial is 0. In other words, the multiplicity of $\alpha$ in the characteristic polynomial (depending on the variable $\lambda$ ) is the maximal nonnegative integer $s$ such that the characteristic polynomial is divisible by $(\lambda - \alpha)^s$.

   Then formulas (2)-(3) can be written as one formula:

$$y_p(t) = t^s e^{\alpha t}(A_0 + A_1 t + A_2 t^2 + \ldots + A_n t^n), \tag{5}$$

   where $s$ is the multiplicity of $\alpha$ in the characteristic polynomial

6. In each of the following equations use the method of undetermined coefficient for finding the general solution of it:

   (a) $y'' - 3y' + 2y = 4e^{3t}$

   (b) $y'' - 3y' + 2y = 4e^{t}$

   (c) $y'' + 10y' + 25y = 3e^{-5t}$

## The case when $g(t) = e^{\alpha t} P_n(t) \cos \omega t$ or $g(t) = e^{\alpha t} P_n(t) \sin \omega t$, where $P_n(t)$ is a polynomial of degree $n$

7. Consider the equation

$$ay'' + by' + cy = e^{\alpha t} P_n(t) \cos \omega t \quad \text{or} \quad ay'' + by' + cy = e^{\alpha t} P_n(t) \sin \omega t \tag{6}$$

In this case the derivatives of a function of the form $e^{\alpha t}Q(t)\cos\omega t$ , where $Q(t)$ is a polynomial, is a sum of functions of the form $e^{\alpha t}Q_1(t)\cos\omega t$ and $e^{\alpha t}Q_2(t)\cos\omega t$, where $Q_1(t)$ and $Q_2(t)$ are polynomials . Therefore, we expect to look for the solutions in the form of sum of such functions. Again the issue of multiplicity is crucial but now we are interested in multiplicity of the complex number

$$\lambda = \alpha + i\omega$$

in the characteristic polynomial. The reason for this is that in fact the considered case is similar to $g(t) = e^{(\alpha+i\omega)t}P_n(t)$, and the latter can be treated similarly to the real case.

For second order equation there are two options for the multiplicity $s$ of $\alpha + i\omega$ in the characteristic polynomial: $s = \_$ or $s = \_$.

Then we look for a particular solution of (6) in the form

$$y_p(t) = t^s e^{\alpha t}(A_0t^n + A_1t^{n-1} + \ldots + A_n)\cos(\omega t) + t^s e^{\alpha t}(B_0t^n + B_1t^{n-1} + \ldots + B_n)\sin(\omega t), \quad (7)$$

8. Find general solution of

$$y'' + 2y' + 5y = 3\sin 2t$$

9. Based on the method of undetermined coefficients determine the form in you will look for $y_p$ (do not find the values of the undetermined coefficients):

| | $ay'' + by' + cy = g(t)$ | $\lambda_1, \lambda_2$ | $\alpha + i\omega$ | $s$ | $y_p(t)$ |
|---|---|---|---|---|---|
| 1 | $y'' = e^t$ | | | | |
| 2 | $y'' - 4y' + 4 = e^{2t}$ | | | | |
| 3 | $y'' + 2y' + 10y = 6\cos(3t)$ | $\lambda_{1,2} = -1 \pm 3i$ | | | |
| 4 | $y'' + 2y' + 10y = 6e^{-t}\sin(3t)$ | $\lambda_{1,2} = -1 \pm 3i$ | | | |
| 5 | $y'' + 2y' + 10y = (t^3 - 1)\sin(3t)$ | $\lambda_{1,2} = -1 \pm 3i$ | | | |
| 6 | $y'' + 2y' + 10y = t^2e^{-t}\cos(3t)$ | $\lambda_{1,2} = -1 \pm 3i$ | | | |
| 7 | $y'' = 2t - 2013$ | $\lambda_1 = \lambda_2 = 0$ | | | |

10.

REMARK 1. *Given an equation*

$$ay'' + by' + cy = g_1(t) + g_2(t) \quad (8)$$

*where* $g_1(t)$ *are* $g_2(t)$ *are of the form of the right-hand sides of* (1) *or/and* (6) *to find a particular solution we separately find particular solutions of*

$$y'' + by' + cy = g_1(t) \text{ and } y'' + by' + cy = g_2(t)$$

*and then add them.*

EXAMPLE 2. *Based on the method of undetermined coefficients determine the form in you will look for $y_p$ (do not find the values of the undetermined coefficients):*

$$y'' - 4y' + 4 = e^{2t} + e^{-2t}.$$

11.

REMARK 3. *(***Enrichment for bonus question 4 of homework 8***) The analog of the method of undetermined coefficients can be applied also to systems of $n$ nonhomogeneous equations of the form:*

$$\mathbf{X}'(t) = A\mathbf{X}(t) + \mathbf{G}(t), \tag{9}$$

*where for simplicity $\mathbf{G}(t) = e^{\alpha t}\mathbf{Z_n}(t)$ with $\alpha$, and $Z_m(t)$ is a vector function with each component being a polynomial of degree not greater than $m$ and such that at least one component is a polynomial of degree equal to $m$ (the case involving factors $\cos\omega t$ and $\sin\omega t$ can be treated similarly) . We discuss only three particular cases:*

*(a) $\alpha$ is not an eigenvalue of $A$. Then the particular solution in the former case can be found in the form*

$$\mathbf{X_p}(t) = e^{\alpha t}\mathbf{V_m}(t)$$

*where $\mathbf{V_m}(t)$ is a vector function with each component being a polynomial of degree not greater than $m$ and such that at least one component is a polynomial of degree equal to $m$ (so the coefficients of all these components are undetermined coefficients. In particular if $m = 0$, i.e. $\mathbf{Z_m}(t) \equiv \mathbf{Z_0}$, then we look for a solution in the form*

$$\mathbf{X_p}(t) = e^{\alpha t}\mathbf{V_0}$$

*for an undetermined vector $\mathbf{V_0}$.*

*(b) $\alpha$ is equal to an eigenvalue of algebraic multiplicity 1 with an eigenvector $\mathbf{v}$. Then the particular solution can be found in the form*

$$\mathbf{X_p}(t) = At^{m+1}e^{\alpha t}\mathbf{v} + e^{\alpha t}\mathbf{V_m}(t)$$

*where $A$ is a(undetermined) constant and $V_m(t)$ is a vector function with each component being a polynomial of degree not greater than $m$ and such that at least one component is a polynomial of degree equal to $m$. In particular if $m = 0$, i.e. $\mathbf{Z_m}(t) \equiv \mathbf{Z_0}$, then we look for a solution in the form*

$$\mathbf{X_p}(t) = Ate^{\alpha t}\mathbf{v} + e^{\alpha t}\mathbf{V_0}$$

*for an undetermined constant $A$ and a vector $\mathbf{V_0}$.*

*(c) Try to generalized it to the case when $\alpha$ equal to a repeated root for $n = 2$.*

# Forced Vibrations (section 3.8)

12. Suppose now we take into consideration an **external force** $F(t)$ acting on a vibrating spring/mass system. The inclusion of $F(t)$ in the formulation of Newton's second law yields

$$mu'' = mg - k(L+u) - \gamma u' + F(t),$$

or taking into account that $mg = kL$ we get the so called ODE of **forced motion**

$$mu'' + \gamma u' + ku = F(t). \tag{10}$$

For this ODE we have the same initial conditions as for unforced vibration. Namely,

$$u(0) = u_0, \quad u'(0) = v_0,$$

where $u_0$ is the initial displacement and $v_0$ is the initial velocity.

## Case of Periodic External Force

13. When $F(t)$ is a periodic function such as

$$F(t) = F_0 \sin(\omega t) \quad \text{or} \quad F(t) = F_0 \cos(\omega t)$$

then by the Method of Undetermined coefficients a particular solution of forced motion can be obtained as

$$u_p(t) = t^s(A\cos(\omega t) + B\sin(\omega t)) = t^s R\cos(\omega t - \delta)$$

Remind that $R = \sqrt{A^2+B^2}$ is amplitude and $\delta$ is phase ($\cos\delta = A/R, \sin\delta = B/R$)

14. Characteristic equation $m\lambda^2 + \gamma\lambda + k = 0$. We consider the case

$$D = \gamma^2 - 4mk < 0,$$

i.e.

$$\gamma < 2\sqrt{km} =: \gamma_{crit}$$

because otherwise there are no unforced oscillations as it follows from the section 3.7.

15. Recall that the roots of the characteristic equation when $D < 0$ are

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \frac{-\gamma \pm i\sqrt{4mk - \gamma^2}}{2m} =$$

$$-\frac{\gamma}{2m} \pm i\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} = \alpha \pm i\sqrt{\omega_0^2 - \alpha^2} = \alpha \pm i\mu,$$

where $\omega_0^2 = \frac{k}{m}$.

16. Solution of the corresponding homogeneous equation is

$$u_h(t) = e^{\alpha t}(C_1 \cos(\mu t) + C_2 \sin(\mu t)) = R_1 e^{\alpha t} \cos(\mu t - \delta),$$

where $R_1 = \sqrt{C_1^2 + C_2^2}$ and $\cos\delta = C_1/R_1, \sin\delta = C_2/R_1$.

17. The general solution of (10) is $u(t) = u_p(t) + u_h(t)$, or

$$u(t) = t^s R\cos(\omega t - \delta) + R_1 e^{\lambda t}\cos(\mu t - \delta). \tag{11}$$

## Forced Damped Vibration

### Steady-State and Transient solutions

18. Motion with damping means $\gamma \neq 0$. It implies the following

- $\gamma$ is a positive constant and then $\alpha = -\frac{\gamma}{2m} < 0$.
- $\lambda_{1,2} \neq \omega$, hence, $s = 0$.
- The general solution of (10) in this case will be

$$u(t) = \underbrace{R\cos(\omega t - \delta)}_{\text{steady-state solution}} + \underbrace{R_1 e^{\lambda t}\cos(\mu t - \delta)}_{\text{transient solution}, u_c(t)}. \tag{12}$$

Emphasize, that transient solution $u_c(t)$ dies of as time increases (because $\alpha < 0$), i.e.

$$\lim_{t\to\infty} u_c(t) = 0.$$

Thus, for large values of $t$, the displacements of mass are closely approximated by $u_p(t)$:

$$u(t) \approx R\cos(\omega t - \delta).$$

## Forced Undamped Vibration

19. Motion without damping means $\gamma = 0$ and then we have the following IVP:

$$mu'' + ku = F(t), \quad u(0) = u_0, \quad u'(0) = v_0,$$

or

$$u'' + \omega_0^2 u = \frac{F(t)}{m}, \quad u(0) = u_0, \quad u'(0) = v_0.$$

20. Consider the particular case of a periodic external force:

$$u'' + \omega_0^2 u = F_0 \cos(\omega t).$$

21. In this case $\alpha = -\frac{\gamma}{2m} = 0, \quad \mu = \sqrt{\omega_0^2 - \lambda^2} = \omega_0$. Thus, general solution is

$$u(t) = Rt^s \cos(\omega t - \delta) + R_1 \cos(\omega_0 t - \delta).$$

- Case 1: $\omega = \omega_0$, i.e. $s = 1$.

22. This is the case when the frequency of the external force coincides with the natural frequency of the system.

23. General solution in this case

$$u(t) = Rt \cos(\omega t - \delta) + R_1 \cos(\omega_0 t - \delta)$$

It follows that $u(t)$ is unbounded as time increases (This phenomenon is known as **pure resonance**.)

- Case 2: $\omega \neq \omega_0$, i.e. $s = 0$.

24. This is the case when the frequency of the external force does not coincide with the natural frequency of the system.

25. Consider the following particular case (when mass is initially at rest):

$$u'' + \omega_0^2 u = F_0 \cos(\omega t), \quad u(0) = 0, \quad u'(0) = 0.$$

26. One can show (see your homework) that the general solution of this ODE is

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left(\cos(\omega t) - \cos(\omega_0 t)\right).$$

Using trigonometric identity

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha - \beta}{2}\sin\frac{\alpha + \beta}{2}$$

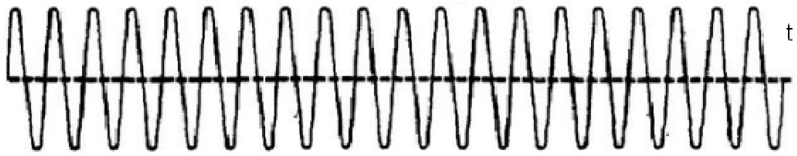
we rewrite the general solution as

$$u(t) = \underbrace{\left[-\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega - \omega_0}{2} t\right)\right]}_{\text{slowly varying sinusoidal amplitude}} \sin\left(\frac{\omega + \omega_0}{2} t\right)$$

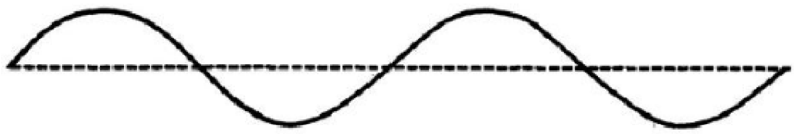
If $|\omega - \omega_0|$ is small, then $\omega + \omega_0$ is much greater than $|\omega - \omega_0|$. Consequently, $\sin\left(\frac{\omega+\omega_0}{2} t\right)$ is rapidly oscillation function comparing to $\sin\left(\frac{\omega-\omega_0}{2} t\right)$ with a slowly varying sinusoidal amplitude $\frac{2F_0}{m|\omega_0^2 - \omega^2|} \sin\left(\frac{\omega-\omega_0}{2} t\right)$.

27. Type of motion processing *a periodic variation of amplitude* is called an amplitude modulation effect (AM).

carrier $\sin\left(\frac{\omega+\omega_0}{2}t\right)$




signal $-\frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\left(\frac{\omega-\omega_0}{2}t\right)$



amplitude modulated wave $u(t)=\left[-\frac{2F_0}{m(\omega_0^2-\omega^2)}\sin\left(\frac{\omega-\omega_0}{2}t\right)\right]\sin\left(\frac{\omega+\omega_0}{2}t\right)$