22: Definition of the Laplace Transform and Solutions of IVP using it (sections 6.1-6.2)

1. Remind the Improper Integral (type I):

$$\int_0^\infty \phi(t)\mathrm{d}t = \lim_{A\to\infty}\int_0^A \phi(t)\mathrm{d}t$$

2. DEFINITION of LAPLACE TRANSFORM Let $f(t)$ be a function defined for $t \geq 0$. Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t \tag{1}$$

is said to be the **Laplace Transform** of f, provided that the integral converges. Note that when the integral (1) converges the result is a function of s.

`Below we use a lowercase letter to denote the function being transformed and the corresponding capital letter to denote its Laplace Transform:`

$$\mathcal{L}\{f(t)\} = F(s), \quad \mathcal{L}\{g(t)\} = G(s), \quad \mathcal{L}\{y(t)\} = Y(s), \text{etc.}$$

3. Example:*Apply the above definition to evaluate Laplace Transform of the following functions:*

   (a) $f(t) = 1$

   (b) $f(t) = e^{at}$

4. $\mathcal{L}$ is a linear transformation:

$$\mathcal{L}\{\alpha f + \beta g\} = \alpha\mathcal{L}\{f\} + \beta\mathcal{L}\{g\}.$$

5. How Laplace Transform might be useful in solving DE? Key property: Under some natural conditions on a function f we have transform of a derivative

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Illustration: We already know that $y(t) = 10e^{-5t}$ is solution of the IVP:

$$y' + 5y = 0, \quad y(0) = 10.$$

Now solve it using Laplace Transform.

6. Example: *Evaluate Laplace Transform of the following functions:*

   (a) $f(t) = \sin(at)$

   (b) In a similar way one can prove that $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2 + a^2}$

7. Transforms of some basic functions[1]

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots \qquad \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

8. Translation in s property:

$$\mathcal{L}\left\{e^{\alpha t} f(t)\right\} = F(s-\alpha)$$

9. EXAMPLE *Evaluate*

   (a) $\mathcal{L}\{e^{\alpha t}\sin\beta t\}$

   (b) $\mathcal{L}\{e^{\alpha t}\cos\beta t\}$

[1] s is sufficiently restricted to guarantee the convergence of the appropriate Laplace Transform.

10. Laplace transform of the derivative: Under some natural conditions on a function f

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

More generally,

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

11. EXAMPLE *Solve for $Y(s)$, the Laplace transform of the solution $y(t)$ to the given initial value problem:*

$$y'' - 2y' + 5y = -8e^{-t}, \quad y(0) = 2, \quad y'(0) = 12.$$

12. Derivative of Laplace transform:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s).$$

13. EXAMPLE *Evaluate* $\mathcal{L}\{t^n e^{\alpha t}\}$

Solution of Initial Value Problems (sec. 6.2)

1. INVERSE LAPLACE TRANSFORMS: If $F(s)$ represents the Laplace Transform of $f(t)$, i.e. $\mathcal{L}\{f(t)\} = F(s)$, then we say that $f(t)$ is the **inverse Laplace Transform** of $F(s)$ and write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

2. Some Inverse Transforms:

Transform	**Inverse Transform**
$\mathcal{L}\{1\} = \dfrac{1}{s}$	$\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} = 1$
$\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$	$\mathcal{L}^{-1}\left\{\dfrac{1}{s^n}\right\} = \dfrac{t^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \ldots$
$\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$	$\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$
$\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$	$\mathcal{L}^{-1}\left\{\dfrac{a}{s^2+a^2}\right\} = \sin at$
$\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$	$\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$
$\mathcal{L}\left\{t^n e^{\alpha t}\right\} = \dfrac{n!}{(s-\alpha)^{n+1}}$	$\mathcal{L}^{-1}\left\{\dfrac{1}{(s-\alpha)^n}\right\} = \dfrac{t^{n-1}e^{\alpha t}}{(n-1)!}$

See Table on the page 317 in the Textbook (or Appendix 2) for more cases.

3. $\mathcal{L}^{-1}$ is a Linear Transform:

$$\mathcal{L}^{-1}\{\alpha f + \beta g\} = \alpha \mathcal{L}^{-1}\{f\} + \beta \mathcal{L}^{-1}\{g\}.$$

4. Note that if, as in section 21 in the notes, we consider a nonhomogeneous equation

$$ay'' + by' + cy = g(t),$$

where a, b, c are real constants and $g(t)$ involves linear combinations, sums and products of

$$t^m, \quad e^{\alpha t}, \quad \sin(\beta t), \quad \cos(\beta t).$$

After applying the Laplace transform to both sides one can find the Laplace transform $Y(s)$ of the solution $y(t)$ and one gets that $Y(s)$ is a rational function of s (i.e it is a ratio of two polynomials) such that the degree of denominator is greater then the degree of numerator. On the other hand, by the right table above all inverse Laplace transforms are taken from elementary fractions appearing in the partial fraction decomposition.

Therefore **in order to find the inverse Laplace transform of a rational function it is enough to find its partial fraction decomposition.** (See Appendix 1: *Inverse Laplace transform of rational functions using Partial Fraction Decomposition*)

5. Example. *Evaluate*

(a) $\mathcal{L}^{-1}\left\{\dfrac{2s+3}{s^2+5s+6}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{2s^2-3s+5}{(s-3)^2(s+4)}\right\}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{3s+5}{s^2+6s+34}\right\}$

6. Consider the n-th order ODE

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + a_1 y' + a_0 y = g(t)$$

subject to

$$y(0) = \alpha_0, y'(0) = \alpha_1, \dots y^{(n-1)} = \alpha_{n-1}.$$

Note that we know how to solve this IVP using the Method Variation of Parameters and the Method of Undetermined Coefficients (for $g(t) = P_n(t)e^{\alpha t}\cos bt$ or $g(t) = P_n(t)e^{\alpha t}\sin bt$)

7. How to solve the given IVP using Laplace Transform:

Step 1. Apply Laplace Transform to both sides of the given ODE. Use linearity and other Laplace Transform properties together with the initial conditions we obtain an algebraic equation in the s-domain for $Y(s) = \mathcal{L}\{y(t)\}$ instead of the given ODE in the t-domain.

Step 2. Solve for $Y(s)$ the algebraic equation obtained in Step 1.

Step 3. Find the inverse Laplace Transform of $Y(s)$ to get $y(t)$.

8. EXAMPLE *Solve IVP*

$$y'' - 2y' + 5y = -8e^{-t}, \quad y(0) = 2, \quad y'(0) = 12.$$

Note that we already found that

$$Y(s) = \frac{2s^2+10s}{(s+1)(s^2-2s+5)}.$$

9. EXAMPLE *Consider the IVP*

$$y'' + 4y' - 5y = te^t, \quad y(0) = 1, \quad y'(0) = 0. \tag{2}$$

SOLUTION (Main Steps, complete the details to practice yourself): Application of Laplace Transform yields:

$$Y(s) = \frac{s^3 + 2s^2 - 7s + 5}{(s-1)^2(s^2 + 4s - 5)} = \frac{s^3 + 2s^2 - 7s + 5}{(s-1)^3(s+5)} \tag{3}$$

Partial Fraction Decomposition:

$$\frac{s^3 + 2s^2 - 7s + 5}{(s-1)^2(s^2 + 4s - 5)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{s+5}, \tag{4}$$

where

$$A = \frac{181}{216}, \quad B = -\frac{1}{36}, \quad C = \frac{1}{6}, \quad D = \frac{35}{216}.$$

Find the inverse Laplace Transform of $Y(S)$ (use Table (see Appendix 2)):

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{181}{216}e^t - \frac{1}{36}te^t + \frac{1}{12}t^2e^t + \frac{35}{216}e^{-5t}. \tag{5}$$

REMARK 1. *As a matter of fact, the method of Laplace transform give another explanation for the form of the solution What is the general used in the method of undetermined coefficients.*

Appendix 1.

Inverse Laplace transform of rational functions using Partial Fraction Decomposition

Using the Laplace transform for solving linear non-homogeneous differential equation with constant coefficients and the right-hand side $g(t)$ of the form $h(t)e^{\alpha t}\cos\beta t$ or $h(t)e^{\alpha t}\sin\beta t$, where $h(t)$ is a polynomial, one needs on certain step to find the inverse Laplace transform of rational functions $\frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials with $\deg P(s) < \deg Q(s)$.
The latter can be done by means of the partial fraction decomposition that you studied in Calculus 2:
One factors the denominator $Q(s)$ as much as possible, i.e. into linear (may be repeated) and quadratic (may be repeated) factors: each linear factor correspond to a real root of $Q(s)$ and each quadratic factor correspond to a pair of complex conjugate roots of $Q(s)$.

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Each factor in the decomposition of $Q(s)$ gives a contribution of certain type to the partial fraction decomposition of $\frac{P(s)}{Q(s)}$. Below we list these contributions depending on the type of the factor and identify the inverse Laplace transform of these contributions:

Case 1 A non-repeated linear factor $(s-a)$ of $Q(s)$ (corresponding to the root a of $Q(s)$ of multiplicity 1) gives a contribution of the form $\frac{A}{s-a}$. Then $\mathcal{L}^{-1}\left\{\frac{A}{s-a}\right\} = Ae^{at}$;

Case 2 A repeated linear factor $(s-a)^m$ of $Q(s)$ (corresponding to the root a of $Q(s)$ of multiplicity m) gives a contribution which is a sum of terms of the form $\frac{A_i}{(s-a)^i}$, $1 \le i \le m$.
Then $\mathcal{L}^{-1}\left\{\frac{A_i}{(s-a)^i}\right\} = \frac{A_i}{(i-1)!}t^{i-1}e^{at}$;

Case 3 A non-repeated quadratic factor $(s-\alpha)^2+\beta^2$ of $Q(s)$ (corresponding to the pair of complex conjugate roots $\alpha \pm i\beta$ of multiplicity 1) gives a contribution of the form
$\frac{Cs+D}{(s-\alpha)^2+\beta^2}$.
It is more convenient here to represent it in the following way:
$\frac{Cs+D}{(s-\alpha)^2+\beta^2} = \frac{A(s-\alpha)+B\beta}{(s-\alpha)^2+\beta^2}$. Then

$$\mathcal{L}^{-1}\left\{\frac{A(s-\alpha)+B\beta}{(s-\alpha)^2+\beta^2}\right\} = Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t;$$

Case 4 A repeated quadratic factor $\left((s-\alpha)^2+\beta^2\right)^m$ of $Q(s)$ (corresponding to the pair of complex conjugate roots $\alpha \pm i\beta$ of multiplicity m) gives a contribution which is a sum of terms of the form

$$\frac{C_i s + D_i}{\left((s-\alpha)^2+\beta^2\right)^i} = \frac{A_i(s-\alpha)+B_i\beta}{\left((s-\alpha)^2+\beta^2\right)^i},$$

where $1 \leq i \leq m$.

The calculation of the inverse Laplace transform in this case is more involved. It can be done as a combination of the property of the derivative of Laplace transform and the notion of *convolution* that will be discussed in section 6.6 or using decomposition to linear factors using complex roots as in Enrichment 8.

Appendix 2.

(from the textbook, page 317)

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}$, $s > 0$
2. e^{at}	$\frac{1}{s-a}$, $s > a$
3. t^n, $n =$ positive integer	$\frac{n!}{s^{n+1}}$, $s > 0$
4. t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$, $s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}$, $s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}$, $s > \lvert a \rvert$
8. $\cosh at$	$\frac{s}{s^2-a^2}$, $s > \lvert a \rvert$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
11. $t^n e^{at}$, $n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
12. $u_c(t)$	$\frac{e^{-cs}}{s}$, $s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$, $c > 0$
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$