23: Step functions. Differential equations with discontinuous forcing functions (sections 6.3 and 6.4)

1. Consider the n-th order linear ODE with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_1 y' + a_0 y = g(t),$$

where $g(t)$ is a piecewise continuous function (function with jump discontinuities).

Jump discontinuities occur naturally in engineering problems such as electrical circuits with on/off switches. To handle such behavior, Heaviside introduced the following step function.

2. **Unit Step Function** $u_c(t)$ $(c \geq 0)$ is defined by

$$u_c(t) = \begin{cases} 0, & 0 \leq t < c \\ 1, & t \geq c \end{cases}$$

3. When a function $f(t)$ defined for $t \geq 0$ is multiplied by $u_c(t)$, this unit step function "turns off" a portion of the graph of that function. For example, consider $(t^2 + 1)u_3(t)$.

4. *FACT 1.* Any function with jump discontinuities at $t = c_1, c_2, \ldots, c_k$ can be represented in terms of unit step functions. In other words, we can use unit step function to write a piecewise-defined functions in a compact form.

5. Express f in terms of unit step function

(a) $f(t) = \begin{cases} 4, & 0 \leq t < 3 \\ 1, & 3 \leq t < 5 \\ -2, & 5 \leq t \end{cases}$

(b) $f(t) = \begin{cases} f_1(t), & 0 \leq t < c_1 \\ f_2(t), & c_1 \leq t < c_2 \\ f_3(t), & c_2 \leq t \end{cases}$

(c) $f(t) = \begin{cases} 3, & 0 \leq t < 2 \\ 1, & 2 \leq t < 3 \\ t, & 3 \leq t < 5 \\ t^2, & 5 \leq t \end{cases}$

6. *FACT 2.* **Translation in t property** for Laplace Transform: if $F(s) = \mathcal{L}\{f(t)\}$ then

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s).$$

7. Find $\mathcal{L}\{u_c(t)\}$

8. Duality between Laplace transform and its inverse:

Derivative	$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	$\mathcal{L}^{-1}\{F'(s)\} = -tf(t)$
Translation	$\mathcal{L}\{e^{\alpha t}f(t)\} = F(s-\alpha)$	$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$

9. Let $f(t)$ will be the same as in 5(c). Find $\mathcal{L}\{f\}$.

10. Find the inverse Laplace transform of

$$H(s) = \frac{e^{-\frac{\pi}{2}s}}{s^2+9} + \frac{se^{-\frac{3\pi}{2}s}}{s^2+4}$$

11. Let

$$g(t) = \begin{cases} 20, & 0 \leq t < 3\pi, \\ 0, & 3\pi \leq t < 4\pi \\ 20, & 4\pi \leq t \end{cases}$$

(a) Solve IVP:

$$y'' + 2y' + 2y = g(t), \quad y(0) = 10, \quad y'(0) = 0,$$

Solution:

Step 1. Express $g(t)$ in compact form.

Step 2. Find $\mathcal{L}\{g(t)\} = G(s)$.

Step 3. Find $\mathcal{L}\{y'' + 2y' + 2y\}$.

Step 4. Combine steps 2& 3 to get $\mathcal{L}\{y(t)\} = Y(s)$.

Step 5. Apply inverse Laplace transform to find $y(t)$. This step usually requires partial fraction decomposition.

(b) Sketch the graph of $y(t)$.