23: Step functions. Differential equations with discontinuous forcing functions (sections 6.3 and 6.4)

1. Consider the n-th order linear ODE with constant coefficients:

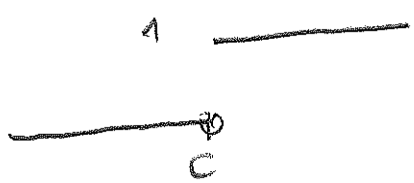
$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_1 y' + a_0 y = g(t),$$

where $g(t)$ is a piecewise continuous function (function with jump discontinuities).

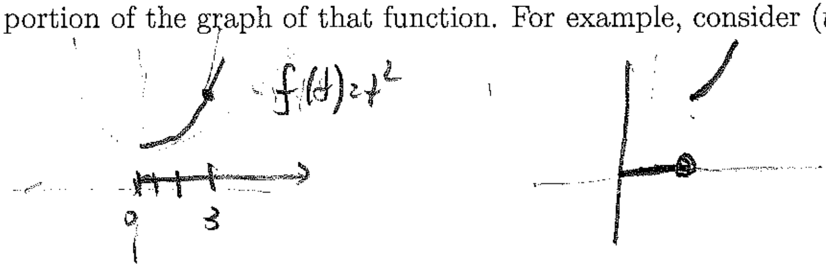
Jump discontinuities occur naturally in engineering problems such as electrical circuits with on/off switches. To handle such behavior, Heaviside introduced the following step function.

2. **Unit Step Function** $u_c(t)$ $(c \geq 0)$ is defined by

$$u_c(t) = \begin{cases} 0, & 0 \leq t < c \\ 1, & t \geq c \end{cases}$$



3. When a function $f(t)$ defined for $t \geq 0$ is multiplied by $u_c(t)$, this unit step function "turns off" a portion of the graph of that function. For example, consider $(t^2 + 1)u_3(t)$.



4. *FACT 1.* Any function with jump discontinuities at $t = c_1, c_2, \ldots, c_k$ can be represented in terms of unit step functions. In other words, we can use unit step function to write a piecewise-defined functions in a compact form.

5. Express f in terms of unit step function

(a) $f(t) = \begin{cases} 4, & 0 \leq t < 3 \\ 1, & 3 \leq t < 5 \\ -2, & 5 \leq t \end{cases}$

$4 + (1-4)u_3(t) + (-2-1)u_5(t)$

(b) $f(t) = \begin{cases} f_1(t), & 0 \le t < c_1 \\ f_2(t), & c_1 \le t < c_2 \\ f_3(t), & c_2 \le t \end{cases}$

$f_1(t) + (f_2(t) - f_1(t))\, u_{c_1}(t) + f_3(t) - f_2(t)$

(c) $f(t) = \begin{cases} 3, & 0 \le t < 2 \\ 1, & 2 \le t < 3 \\ t, & 3 \le t < 5 \\ t^2, & 5 \le t \end{cases}$

$f(t)$ $3 + (1-3)\, u_2(t) + (t-1)\, u_3(t) + (t^2 - t)\, u_5(t) = 3 - 2u_2(t) +$
$(t-1)\, u_3(t) + (t^2 - t)\, u_5(t)$

6. *FACT 2.* **Translation in t property** for Laplace Transform: if $F(s) = \mathcal{L}\{f(t)\}$ then

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s).$$

$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt = \int_c^\infty e^{-st} f(t-c)\, dt =$ $\tau = t - c$, $d\tau = dt$, $t = \tau + c$

$= \int_0^\infty e^{-s(\tau + c)} f(\tau)\, d\tau = e^{-cs} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau = e^{-cs} F(s)$

7. Find $\mathcal{L}\{u_c(t)\}$.

$\mathcal{L}\{u_c(t)\} = \{u_c(t) \cdot 1\} = \frac{e^{-cs}}{s}$
$f(t) = 1$

8. Duality between Laplace transform and its inverse:

Derivative	$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	$\mathcal{L}^{-1}\{F'(s)\} = -tf(t)$
Translation	$\mathcal{L}\{e^{\alpha t}f(t)\} = F(s-\alpha)$	$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$

$\mathcal{L}\{u_c(t) f(t)\} = e^{-cs}\, \mathcal{L}\{f(t+c)\}(s)$

9. Let $f(t)$ will be the same as in 5(c). Find $\mathcal{L}\{f\}$.

1) $\mathcal{L}\{3\} = \frac{3}{s}$

2) $\mathcal{L}\{2u_2(t)\} = 2\,\frac{e^{-2s}}{s}$

3) $\mathcal{L}\{(t-1)u_3(t)\} = e^{-3s}\mathcal{L}\{(\underbrace{t-1+3}_{t+2})\} =$

$$= e^{-3s}\big(\underbrace{\mathcal{L}\{t\}}_{\frac{1}{s^2}} + \underbrace{\mathcal{L}\{2\}}_{\frac{2}{s}}\big)$$

4) $\mathcal{L}\{(t^2-t)u_5(t)\} = e^{-5s}\mathcal{L}\{(t+5)^2-(t+5)\} =$

$$= e^{-5s}\mathcal{L}\{\underbrace{t^2+10t+25-t-5}_{t^2+9t+20}\} = e^{-5s}\left(\frac{2}{s^3}+\frac{9}{s^2}+\frac{20}{s}\right)$$

$\Downarrow$

$$\mathcal{L}\{f(t)\} = \frac{3}{s}+\frac{2e^{-2s}}{s} + e^{-3s}\left(\frac{1}{s^2}+\frac{2}{s}\right) + e^{-5s}\left(\frac{2}{s^3}+\frac{9}{s^2}+\frac{20}{s}\right)$$

10. Find the inverse Laplace transform of

$$H(s) = \frac{e^{-\frac{\pi}{2}s}}{s^2+9} + \frac{se^{-\frac{3\pi}{2}s}}{s^2+4}$$

$\mathcal{L}^{-1}\left\{\frac{e^{-cs}}{s^2+9}\right\}$ $\quad \mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \frac{1}{3}\sin 3t \Rightarrow$

$\beta = 3$

$\mathcal{L}^{-1}\left\{\frac{e^{-cs}}{s^2+9}\right\} = \frac{1}{3}u_{\frac{\pi}{2}}(t)\,\sin 3\left(t-\frac{\pi}{2}\right) =$

$= \frac{1}{3}u_{\frac{\pi}{2}}(t)\,\underbrace{\sin\left(3t-\frac{3\pi}{2}\right)}_{\cos 3t}$

$-\frac{3\pi}{2}$ $+$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$

$\beta = 2$

$\mathcal{L}^{-1}\left\{e^{-\frac{3\pi}{2}s}\frac{s}{s^2+4}\right\} = u_{\frac{3\pi}{2}}(t)\,\overbrace{\cos\left(2\left(t-\frac{3\pi}{2}\right)\right)}^{-\cos 2t}$

$(2t-3\pi)$

-3π

$$\boxed{\frac{1}{3}u_{\frac{\pi}{2}}(t)\cos 3t - u_{\frac{3\pi}{2}}(t)\cos 2t}$$

11. Let

$$g(t) = \begin{cases} 20, & 0 \le t < 3\pi, \\ 0, & 3\pi \le t < 4\pi \\ 20, & 4\pi \le t \end{cases}$$

(a) Solve IVP:

$$y'' + 2y' + 2y = g(t), \quad y(0) = 10, \quad y'(0) = 0,$$

Solution:

Step 1. Express $g(t)$ in compact form.

$$20 + (0-20)u_{3\pi}(t) + 20\, u_{4\pi}(t) =$$
$$= 20 - 20\, u_{3\pi}(t) + 20\, u_{4\pi}(t)$$

Step 2. Find $\mathcal{L}\{g(t)\} = G(s)$.

$$\mathcal{L}\{g(t)\} = \frac{20}{s} - \frac{20}{s}e^{-3\pi s} + \frac{20}{s}e^{-4\pi s}$$

Step 3. Find $\mathcal{L}\{y'' + 2y' + 2y\}$.

$$2 \times \mathcal{L}\{y\} = Y$$
$$+\ 2 \times \mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 10$$
$$+\ 1 \times \mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0) = s^2 Y(s) - 10s$$

$$\mathcal{L}\{y'' + 2y' + 2y\} = (s^2 + 2s + 2)Y(s) - 10s - 20$$

Step 4. Combine steps 2& 3 to get $\mathcal{L}\{y(t)\} = Y(s)$.

$$(s^2 + 2s + 2)\,Y(s) - 10s - 20 = \frac{20}{s} - \frac{20}{s}e^{-3\pi s} + \frac{20}{s}e^{-4\pi s}$$

$$\Rightarrow Y(s) = \frac{10s + 20}{s^2 + 2s + 2} + \frac{20}{s(s^2 + 2s + 2)} - \frac{20}{s(s^2 + 2s + 2)}e^{-3\pi s} +$$

$$\frac{20}{s(s^2 + 2s + 2)}e^{-4\pi s} = \underbrace{\frac{10s^2 + 20s + 20}{s(s^2 + 2s + 2)}}_{\frac{10}{s}} + \frac{20}{s(s^2 + 2s + 2)}\left(e^{-4\pi s} - e^{-3\pi s}\right) = \boxed{\frac{10}{s} + \frac{20}{s(s^2 + 2s + 2)}\left(e^{-4\pi s} - e^{-3\pi s}\right)}$$

Step 5. Apply inverse Laplace transform to find $y(t)$. This step usually requires partial fraction decomposition.

$$\boxed{y(t) = 10 + 10\,u_{4\pi}(t)\left(1 - e^{4\pi}e^{-t} \times (\cos t + \sin t)\right) - 10\,u_{3\pi}(t)\left(1 + e^{3\pi}e^{-t} \times (\cos t + \sin t)\right)}$$

$$\mathcal{L}^{-1}\left\{\frac{10}{s}\right\} = 10$$

Find $\mathcal{L}^{-1}\left\{\frac{20}{s(\underbrace{s^2+2s+2}_{(s+1)^2+1})}\right\}$

$$\frac{20}{s(s^2+2s+2)} = \frac{A}{s} + \frac{B(s+1)+C}{(s+1)^2+1}$$

$$20 = A\left((s+1)^2+1\right) + \left(B(s+1)+C\right)s$$

To find A plug $s=0$: $20 = A\cdot 2 \Rightarrow A = 10$

Plug $s=-1$: $20 = A - C = 10 - C \Rightarrow C = -10$

Equating coefficients of s^2:

$$0 = A + B = 10 + B \Rightarrow B = -10 \Rightarrow$$

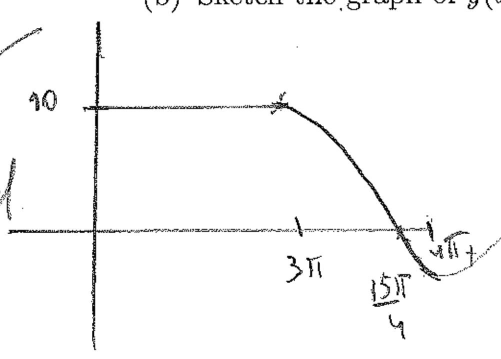
$$\frac{20}{s(s^2+2s+2)} = 10\left(\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+1}\right) \Rightarrow \mathcal{L}^{-1}\left\{\frac{20}{s(s^2+2s+2)}\right\} = 10\left(1 - e^{-t}(\cos t + \sin t)\right)$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{20}{s(s^2+2s+2)}\left(e^{-4\pi s} - e^{-3\pi s}\right)\right) = 10\,u_{4\pi}(t)\left(1 - e^{-t+4\pi}\Big(\underbrace{\cos(t-4\pi)}_{\cos t} + \underbrace{\sin(t-4\pi)}_{\sin t}\Big)\right) -$$

$$-10\,u_{3\pi}(t)\left(1 - e^{-t+3\pi}\Big(\underbrace{\cos(t-3\pi)}_{-\cos t} + \underbrace{\sin(t-3\pi)}_{-\sin t}\Big)\right)$$

(b) Sketch the graph of $y(t)$.

Can be skipped



$$y(t) = \begin{cases} 10, & 0 \le t < 3\pi \\ -10e^{3\pi}e^{-t}\big(\overbrace{\cos t + \sin t}^{\sqrt{2}\cos(t-\frac{\pi}{4})}\big), & 3\pi \le t < 4\pi \\ 10\left(1 - \left(e^{4\pi} + e^{3\pi}\right)e^{-t}\big(\underbrace{\cos t + \sin t}_{\sqrt{2}\cos(t-\frac{\pi}{4})}\big)\right), & 4\pi \le t \end{cases}$$