24: Impulse Function and convolution integral (sections 6.5 and 6.6)

Impulse function

1. In applications (mechanical systems, electrical circuits etc) one encounters functions (external force) of large magnitude that acts only for a very short period of time. To model violent forces of short duration the so called delta function is used. This function was introduces by Paul Dirac.

2. If a force $F(t)$ acts on a body of mass m on the time interval $[t_0, t_1]$, then the impulse due to F is defined by the integral

$$\text{impulse} = \int_{t_0}^{t_1} F(t)\mathrm{d}t = \int_{t_0}^{t_1} ma(t)\mathrm{d}t = \int_{t_0}^{t_1} m\frac{\mathrm{d}v(t)}{\mathrm{d}t}\mathrm{d}t = mv(t_1) - mv(t_0)$$

3. The impulse equals the change in momentum that the force F imparts in the time interval $[t_0, t_1]$.

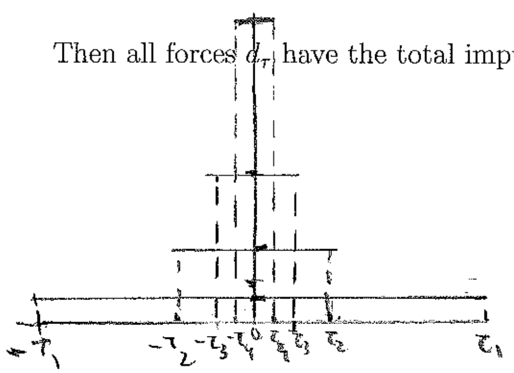
   When a hammer strikes an object, it transfers momentum to the object. This change in momentum takes place over a very short period of time. The change in momentum (=the impulse) is the area under the curve defined by $F(t)$

$$\text{the total impulse of the force } F(t) = \int_{-\infty}^{\infty} F(t)\mathrm{d}t$$

4. Consider a family of piecewise functions (forces)

$$d_\tau = \begin{cases} \dfrac{1}{2\tau}, & \text{if } |t| < \tau \\ 0, & \text{if } |t| \geq \tau. \end{cases}$$

   Then all forces d_τ have the total impulse which is equal 1.



5. Dirac DELTA Function: As $\tau \to 0$ the force acts on infinitesimally small interval, but the total impulse remains 1 (and the magnitude of the force becomes infinitely large). The *idealized unit impulse force* $\delta(t)$ is the force concentrated at $t = 0$ with total impulse one, and it should be understand as a certain limit of the functions $\delta_\tau(t)$ as $\tau \to 0$.

6. **A bit more rigorous mathematical treatment** Note that $\delta(t)$ is not a function in the usual sense, because

   - Concentration at 0 means that $\delta(t) = 0$ for $t \neq 0$;
   - Total unit impulse means that $\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1$.

   An ordinary function cannot satisfy both of these two conditions simultaneously. *Delta*-function was introduced by a Physicist **Paul Dirac** in 1930, but a rigorous mathematical treatment of it was developed much later by a Mathematician **Laurent Schwartz** in 1951, who introduced *generalized functions*. According to this δ-function is a so-called linear functional on a certain space of functions assigning to each function f its value at 0,

   $$\delta(f) := f(0).$$

   The actual meaning of the limit $\lim_{\tau\to 0} d_\tau = \delta$ is that for any continuous function f

   $$\lim_{\tau\to 0} \int_{-\infty}^{+\infty} d_\tau(t) f(t)\, dt = f(0) = \delta(f). \tag{1}$$

   This can be easily shown using the Mean Value Theorem.

7. By analogy with the left-hand side of (1) we can formally write that

   $$\int_{-\infty}^{\infty} \delta(t) f(t)\mathrm{d}t = f(0) \tag{2}$$

   for any continuous function f. From now one we will make formal manipulations with this formula, which will lead to true solutions of differential equations, but a rigorous justification of the validity of this manipulations is beyond your current mathematical background.

8. A unit impulse concentrated at $t = t_0$ is denoted by $\delta(t - t_0)$ and by analogy with (3)

   $$\int_{-\infty}^{\infty} \delta(t - t_0) f(t)\mathrm{d}t = f(t_0), \quad t \neq t_0. \tag{3}$$

9. Laplace Transform of *delta*-function:

   (a) $\mathcal{L}\{\delta(t)\} =$ $\int_0^\infty \delta(t)e^{-st}dt = \int_{-\infty}^{\infty} \delta(t)\underbrace{e^{-st}}_{f(t)}dt = f(0) = e^0 = 1$

   (b) For $t_0 \geq 0$
   $\mathcal{L}\{\delta(t - t_0)\} =$ $\int_0^\infty \delta(t)e^{-st}dt = \int_{-\infty}^{\infty} \delta(t)\underbrace{e^{-st}}_{f(t)}dt = f(t_0) = e^{-t_0 s}$

Convolution Integral (section 6.6)

12. If f and g are piecewise continuous on $[0,\infty)$, then the **convolution**, $f*g$, is defined by the integral
$$f*g=\int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau.$$

13. Convolution is commutative, i.e. $f*g=g*f$

14. Convolution Theorem. If $F(s)=\mathcal{L}\{f(t)\}$ and $G(s)=\mathcal{L}\{G(t)\}$ exist for $s\geq a>0$ then for $s>a$
$$\mathcal{L}\{f*g\}=\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}=F(s)G(s),$$
or
(the proof will be skipped)
$$\mathcal{L}^{-1}\{F(s)G(s)\}=f*g.$$

15. Use the convolution integral to compute

(a) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-a)(s-b)}\right\}$ (of course this can be done using the partial fraction)

$F(s)=\frac{1}{s-a}$, $G(s)=\frac{1}{s-b}$

$f(t)=e^{at}$ $\quad g(t)=e^{bt}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}=f*g(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau=$$
$$=\int_0^t e^{a(t-\tau)}e^{b\tau}d\tau=e^{at}\int_0^t e^{(b-a)\tau}d\tau=e^{at}\frac{1}{b-a}e^{(b-a)\tau}\Big|_{\tau=0}^{t}$$
$$=e^{at}\frac{e^{(b-a)t}-1}{b-a}=\frac{e^{bt}-e^{at}}{b-a}$$
(via partial fraction decomposition $\frac{1}{(s-a)(s-b)}=\frac{1}{b-a}\left(\frac{1}{s-b}-\frac{1}{s-a}\right)$)

10. Solve the given IVP and sketch the graph of the solution:

$$y'' + y = \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 1.$$

hummer strike at $t = 2\pi$

Solution Apply Laplace transform:

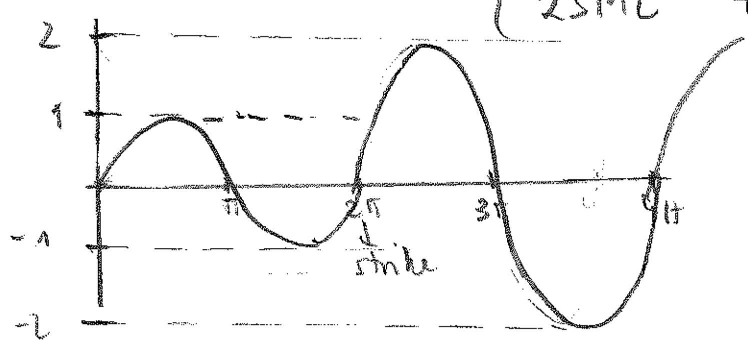
$$\mathcal{L}\{y\} = Y(s)$$
$$\mathcal{L}\{y''\} = s^2 Y(s) - s y(0) - y'(0) = s^2 Y(s) - 1$$
$$\mathcal{L}\{\delta(t-2\pi)\} = e^{-2\pi s}$$

$$\Rightarrow \quad \mathcal{L}\{y'' + y\} = (s^2+1) Y(s) - 1 = e^{-2\pi s} \Rightarrow Y(s) = \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t \Rightarrow \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\} = u_{2\pi}(t) \underbrace{\sin(t - 2\pi)}_{\sin t} = u_{2\pi}(t) \sin t$$

$$\Rightarrow \quad y(t) = \sin t + u_{2\pi}(t) \sin t = \begin{cases} \sin t & 0 \le t < 2\pi \\ 2\sin t & t \ge 2\pi \end{cases}$$

Sketch of the graph



11. Remark:

$$\int_{-\infty}^{t} \delta(t - t_0) \mathrm{d}t = \begin{cases} 0, & t < t_0 \\ 1, & t \ge t_0 \end{cases} = u_{t_0}(t).$$

In other words, derivative of unit step function is *delta*-function.

(b) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$ (for another way to compute using the partial fraction decomposition over complex numbers see Enrichment 8).

Take $F(s)=G(s)=\frac{1}{s^2+1} \Rightarrow f(t)=g(t)=\sin t \Rightarrow$ by the convolution theorem $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}=f*f=\int_0^t \underbrace{\sin(t-\tau)\sin\tau}\,d\tau=$

we use $\sin\alpha\sin\beta=\frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$

here: $\alpha=t-\tau$, $\beta=\tau$

$$=\frac{1}{2}\int_0^t\left(\cos(t-\tau-\tau)-\cos(t-\cancel{\tau}+\cancel{\tau})\right)d\tau=\frac{1}{2}\int_0^t\left(\cos(t-2\tau)\underbrace{-\cos t}_{\text{does not depend on }\tau}\right)d\tau=$$

$$=-\frac{1}{4}\sin(t-2\tau)\Big|_0^t-\frac{1}{2}t\cos t=-\frac{1}{4}\underbrace{\sin(t-2t)}_{-\sin t}+\frac{1}{4}\sin t-\frac{1}{2}t\cos t=\boxed{\frac{1}{2}\sin t-\frac{1}{2}t\cos t}$$

16. Consider IVP:

$$y''+\omega^2y=g(t),\quad y(0)=0,\quad y'(0)=1.$$

(a) Express the solution of the given IVP in terms of the convolution integral.

Apply Laplace: $\mathcal{L}\{y\}=Y$

$\mathcal{L}\{y''\}=s^2Y-s\underset{0}{y(0)}-\underset{1}{y'(0)}=s^2Y-1$

$$\Rightarrow \mathcal{L}\{y''+\omega^2y\}=(s^2+\omega^2)Y-1=G(s)\Rightarrow$$

$$Y=\frac{1}{s^2+\omega^2}G(s)+\frac{1}{s^2+\omega^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\}=\frac{1}{\omega}\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\}=\frac{1}{\omega}\sin\omega t\Rightarrow$$

$$\mathcal{L}^{-1}(Y)=\frac{1}{\omega}\sin\omega t*g(t)+\frac{1}{\omega}\sin\omega t=$$

$$=\frac{1}{\omega}\int_0^t\sin\omega(t-\tau)g(\tau)\,d\tau+\frac{1}{\omega}\sin\omega t$$

(b) Use the Methof of Variation of Parameters to solve the given IVP and compare the result with (a).

Homogeneous equation: $y''+\omega^2 y=0 \Rightarrow$ $\cos\omega t$ & $\sin\omega t$ is a fundamental set of solutions. Look for a solution of our nonhomogeneous equation in the form $y(t)=u_1(t)\cos\omega t+u_2(t)\sin\omega t$ with

$$\begin{pmatrix}\cos\omega t & \sin\omega t\\ -\omega\sin\omega t & \omega\cos\omega t\end{pmatrix}\begin{pmatrix}u_1'\\ u_2'\end{pmatrix}=\begin{pmatrix}0\\ g(t)\end{pmatrix} \Rightarrow u_1'=\frac{\begin{vmatrix}0 & \sin\omega t\\ g & \omega\cos\omega t\end{vmatrix}}{\begin{vmatrix}\cos\omega t & \sin\omega t\\ -\omega\sin\omega t & \omega\cos\omega t\end{vmatrix}}=-\frac{1}{\omega}\sin\omega t\, g(t)$$

$$\Rightarrow u_1(t)=-\frac{1}{\omega}\int_0^t \sin\omega\tau\, g(\tau), \quad u_2'=\frac{1}{\omega}g(t)\cos\omega t \Rightarrow u_2(t)=\frac{1}{\omega}\int_0^t g(\tau)\cos\omega\tau$$

$$\Rightarrow y(t)=\frac{1}{\omega}\int_0^t(\underbrace{-\sin\omega\tau\cos\omega t+\cos\omega\tau\sin\omega t}_{\sin\omega(t-\tau)})g(\tau)d\tau+C_1\cos\omega t+C_2\sin\omega t$$

Plugging initial condition we get $C_1=0$, $C_2=\frac{1}{\omega}$ and so the results match

The impulse response of the sysem

17. Given linear IVP of the second order

$$ay''+by'+cy=g(t), \quad y(0)=y_0, y'(0)=y_1 \tag{4}$$

(here we consider an equation of second order for simplicity only, the same will work for higher order equations).

Applying Laplace transform we get $\mathcal{L}\{y\}=Y$, $\mathcal{L}\{y'\}=sY-y_0$

$\mathcal{L}\{y''\}=s^2Y(s)-sy_0-y_1 \Rightarrow$

$\mathcal{L}\{ay''+by'+cy\}=(as^2+bs+c)Y(s)-((as+b)y_0+ay_1)=G(s)$

18. Let

$$H(s)=\frac{1}{as^2+bs+c}. \tag{5}$$

and $P(s)=(as+b)y_0+ay_1$ $\Rightarrow$ $Y(s)=H(s)G(s)+H(s)P(s)$

Applying the inverse Laplace transform and using the Convolution Theorem we get:

$$y(t) = h * g + h * p \tag{6}$$

(a) The first term is the solution of the nonhomogeneous IVP with zero initial conditions:

$$ay'' + by' + cy = g(t), y(0) = 0; y'(0) = 0 \tag{7}$$

(b) The second term is the solution of homogeneous equation $ay'' + by' + cy = 0$ but with the same initial conditions as in (4).

In particular, if the initial values are zero as in (7), then $P(s) = 0$ and (6) implies that

$$y(t) = h * g = \int_0^t h(t - \tau)g(\tau)\mathrm{d}\tau. \tag{8}$$

19. **What is the meaning of $h(t)$?** It is the solution of IVP

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, y'(0) = 0 \tag{9}$$

and therefore it is called the *impulse response* of the system (describing how the system responses to the unit impulse force applied at time $t = 0$)

20. **Heuristic explanation of the formula** (8)

Take a partition $0 < t_1 < \ldots < t_n < t_{n+1} = t$ of the interval $[0, t]$ and approximate an external force by a superposition of impulse forces:

$$g(t) \sim \sum_{i=0}^{n} g(t_i)\Delta t_i \delta_(t - t_i),$$

where $\Delta t_i := t_{i+1} - t_i$. Then by the superposition principle

$$f(t) \sim \sum_{i=0}^{n} g(t_i)h(t - t_i)\Delta t_i$$

As Δt_i tends to zero we get

$$f(t) = \int_0^t h(t - \tau)g(\tau)\mathrm{d}\tau$$