TAMU Math Circle, Review of the lesson on March 24 2012
Invariants, coloring, and graphs
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Below is the summary of what we covered in the last Math Circle of March 24.

1. Warming up: Necessary versus sufficient

1. For each of the following sentences replace . . . by one of the following: “necessary”, “sufficient”, or ”necessary and sufficient”:

(a) In order that the sum of two integer numbers will be even it is . . . that each of these numbers is even;

   **Answer: Sufficient (but not necessary).**
   It is sufficient: If two numbers are even then their sum is even
   It is not necessary: If the sum of two integer is even then they are not necessary both even.
   They can be also both odd.

(b) In order that a number will be divisible by 15 it is . . . that it is divisible by 5;

   **Answer: Necessary (but not sufficient).**
   It is necessary: If a number is divisible by 15 then it is divisible by 5.
   It is not sufficient: if a number is divisible by 5 but not divisible by 3 (for example, 10) then it is not divisible by 15.

(c) In order that a number will be divisible by 10 it is . . . that it is divisible by 2 and 5;

   **Answer: Necessary and Sufficient**

(d) In order that the sum of two numbers will be greater than 30 it is . . . that at least one of these numbers is greater than 15;

   **Answer: Necessary (but not sufficient).**
   It is necessary: If the sum of two numbers is greater than 30 then at least one of them must be greater than 15. Indeed, assume by contradiction that both of these numbers are not greater than 15, then their sum is not greater than 30 ⇒ contradiction.
   It is not sufficient. Counterexample: take 16 and 1, then 16 > 15 but 16 + 1 ≤ 30

(e) In order that a quadrilateral will be a rectangle it is . . . its diagonals are equal.

   **Answer: Necessary (but not sufficient).**
   It is necessary: If a quadrilateral is a rectangle then its diagonals are equal (for those whi studied/studies Geometry: use one of the property of congruent triangles for the proof)
   It is not sufficient: take an isosceles trapezoid: its diagonals are equal but it is not a rectangle.
2. Problems using Invariants

2.1 Miscellaneous

2. (a) There is only two letters in the alphabet of the Ao-Ao language: $A$ and $O$. Moreover, the language satisfies the following rules: if you delete two neighboring letters $AO$ from any word, then you will get the word with the same meaning. Similarly the meaning of a word will not change if you insert the combination $OA$ or $AAOO$ in any place in the word. Can we be sure that $AOO$ and $OAA$ have the same meaning?

*Hint:* What quantity is preserved when we make a change of a word that does not change its meaning?

**Answer:** No, we cannot be sure they have the same meaning.

If we delete $AO$ then the number of $A$ decreases by one and the number of $O$ decreases by 1. Therefore the quantity $#A - #O$ is preserved (does not change). The same conclusion can be made if we do other permitted changes which do not change the meaning of words, namely the quantity $#A - #O$ does not change under any permitted change of a word that does not change its meaning. One says that $#A - #O$ is an invariant under the permitted changes of words.

So, if for two words this quantity is different, we cannot obtain one of this words from the other one by a permitted change. So, we cannot be sure they have the same meaning. In our case for the word $AOO$ the quantity $#A - #O = 1 - 2 = -1$ and for the word $OAA$ the quantity $#A - #O = 2 - 1 = 1$, so this quantities are not equal and we cannot be sure they have the same meaning.

(b) A word in the Ao-Ao language is called **minimal** if it contains the minimal number of letters among all words with the same meaning (in other words, it has the minimal length among all words with the same meaning). Describe all possible words in the Ao-Ao language. Here we assume in addition that any combination of letters in the Ao-Ao has some meaning.

**Answer:** All minimal words are words among the following one: $AO$, $OA$, the words containing only letters $A$, namely the words of the type $AA\ldots A$, and the words containing only letter $O$, namely the words of the type $OO\ldots O$. Note that in general not all of the words in this list are necessary minimal, because some of them, of different length, may have the same meaning.

Indeed, under the additional assumption if one deletes two neighboring letters $OA$ from any word then one gets a word with the same meaning, because the first one is obtained from the second one by inserting $OA$ which is a permitted operation. Note that without the additional assumption this argument is not true, because removing $OA$ we may get a meaningless combination of letters.
So, if the word has both letters A and O and it is not equal to one of the words AO and OA then it cannot be minimal, because at some place the word will contain a piece OA or AO (i.e. at some place the letters A and O are neighboring) and the word can be shorten (by removing OA or AO) without changing its meaning. The words AO and OA are special because by removing AO and OA we get nothing (an empty word, which has no meaning).

3. The numbers 1, 2, 3, …, 19, 20 are written on a blackboard. It is allowed to erase any two numbers a and b and write the new number a + b − 1 (in other words, it is allowed to erase any two numbers and write a new number which is the sum of the erased numbers decreased by 1). What number will be on the blackboard after 19 such operations?

Remark Note that this problem can be solved without finding invariants. Nevertheless please try to find an invariant in this problem.

Answer: 191

One important remark before going to the proof: It is not clear from the beginning whether we will always get the same number at the end independently what pairs of numbers a and b we erase and replace by a + b − 1 on each step. Some of you guessed the right answer by choosing a particular strategy but this cannot be considered as a right solution of the problem because you have to justify that your answer is independent of the chosen strategy.

Solution not using invariants: Let S be a sum of all numbers: S = 1 + 2 + … + 20 (later on we will show that S = 210 but it is not important right now). Then no matter what pair of numbers we choose on the first step the sum of the new 19 numbers obtained on a blackboard after implementing our permitted operation will be 1 less, namely S − 1 (note that only two numbers are involved in our operation and their sum after the operation is decreased by 1, therefore the total sum of all numbers is decreased by 1). Then after applying a permitted operation for the second time the total sum of the new 18 numbers on a blackboard becomes equal to S − 2 and so on. After 19 operation we are left with one number and it will be equal to S − 19 (= 210 − 19 = 191).

Solution using invariants: Note that after applying one operation the total sum of the numbers on a blackboard is decreased by 1 and the number of the numbers on the blackboard is decreased by 1 as well. Therefore the following quantity is preserved after any operation:

(The sum of the numbers on a blackboard − the number of the numbers on the blackboard).

In other words this quantity is an invariant under any permitted operation.

At the beginning the quantity (the sum of the numbers on a blackboard − the number of the numbers on the blackboard) is equal to S − 20 (we have 20 numbers). Assume that at the end the number a is remained. Then at the end the quantity (the sum of the numbers on a blackboard − the number of the numbers on the blackboard) is equal to a − 1 (we have only one number). Since the quantity is preserved under any permitted operation it must remain the same at the beginning and at the end, namely S − 20 = a − 1. Therefore a = S − 19 = 191.
By the way, do you know how to calculate effectively the sum of the first $n$ natural numbers, i.e. $1 + 2 + 3 + \ldots + n$?

For example, to find $S = 1 + 2 + \ldots + 20$ do we need to make 19 additions? No!

Write our numbers (with the numbers in the increasing order) in one row, then reverse the order and write the same numbers in the reversed order in the second row as follows:

\[
\begin{array}{ccccccc}
1 & 2 & 3 & \ldots & 19 & 20 \\
20 & 19 & 18 & \ldots & 2 & 1
\end{array}
\]

Then the sum of the numbers in each column is the same and equal to 21, the number of columns is equal to 20. So, on one hand, the total sum of numbers in the table is equal to $20 \cdot 21$ and, on the other hand, it is twice of the sum we are looking for, namely $2S$. Therefore, $S = \frac{20 \cdot 21}{2} = 10 \cdot 21 = 210$.

In completely the same manner you can prove that $1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}$.

4. The numbers 1, 2, 3, \ldots, 19, 20 are written on a blackboard. It is allowed to erase any two numbers $a$ and $b$ and write the new number $ab + a + b$. Which number can be on the blackboard after 19 such operations?

**Hint:** Note that $(a + 1)(b + 1) = (ab + a + b) + 1$. What quantity is preserved?

This problem was not discussed in detail. Please continue to think about it. The hint is important.

You may use the following terminology: The product $1 \cdot 2 \cdot 3 \ldots \cdot n$ of the first $n$ natural numbers is called $n$ factorial and it is denoted by $n!$, For example $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$, $5! = 120$, $6! = 720$. Note that by convention $0! = 1$.

**2.2 Remainders modulo natural numbers as invariants**

5. The numbers 1, 2, \ldots, $n$ are written on a blackboard. It is permitted to erase any two of them and replace them with their differences. Can this operation be used to obtain a situation where all the numbers on the blackboard are zeros? How the answer depends on $n$?

**Directions** First make experiments for several small $n$, then make a conjecture based on your experiments, then justify your conjecture.

**Hint:** The problem has an invariant.

This problem was not discussed. Please continue to think about it.
3. Problem using coloring

3.1 Chess board coloring again

6. Prove that a $10 \times 10$ board cannot be covered without overlapping by the following shape:

This problem was discussed in the intermediate school students group but not in the middle school students group. The middle school student please continue to think about this. The chessboard coloring is important. It gives you the shapes of two different types. Use this fact.

3.2 Other type of coloring

7. Remove a single square from an $8 \times 8$ chessboard. Can we cover the remaining board with twenty one $1 \times 3$ polyminos (straight trominoes)?

This problem was discussed in the intermediate school students group but not in the middle school students group. The middle school student please continue to think about this. Use the coloring by three colors such that each straight tromino covers the three squares with three different colors. Then use certain symmetry of the problem to get another coloring. This will allow you to restrict your possibilities significantly.

8. Prove that $8 \times 8$ chessboard cannot be covered (without overlapping) by fifteen $1 \times 4$ polyminos and the single L-shaped polymino shown in the figure:

This problem was not discussed. Try to use some coloring of the board by 2 colors which distinguish $1 \times 4$ polyminos and the single L-shaped polymino.

4. Graphs

4.1 Connected graphs

9. In some country there are 15 towns each of which is connected to at least 7 others. Prove that one can travel from any town to any other town, possibly passing through some towns between.

This problem was discussed in the middle school students group but was not discussed in the intermediate school students group. The younger students, please try to think about it.

4.2 Degree of vertex
10. In Smallville there are 17 telephones. Can they be connected by wires so that each telephone is connected with exactly five phones?

This problem was discussed in the middle school students group at the very end but was not discussed in the intermediate school students group. The younger students, please try to think about it. We will repeat it in both groups.

The rest was not discussed. Please try it by yourself.

4.3 Seven bridges of Königsberg: unicursal (Eulerian) graph

The city of Königsberg in East Prussia (a German Kingdom in 18-19 century, now this city is called Kaliningrad and it is a part of Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges as shown in the figure below.

11. The famous problem solved by Leonard Euler in 1735: Is it possible to find a walk through the city that would cross each bridge once and only once?
12. A group of islands are connected by bridges in such a way that one can walk from any island to any other. A tourist walked around every island, crossing each bridge exactly once. H visited the Emerald island three times. How many bridges are there to the Emerald Island if

(a) the tourist neither started nor ended on the Emerald Island;
(b) the tourist started on the Emerald Island, but did not end there;
(c) the tourist started and ended on the Emerald Island.