Main properties of Laplace transform

**Definition 1.** (piecewise continuity) A function $f$ is called piecewise continuous on the interval $a \leq t \leq b$ if there exists a partition of this interval by finite number of points $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ such that

1. $f$ is continuous on each open subinterval $t_{i-1} < t < t_i$,
2. one-sided limits $\lim_{t \to t_i^+} f(t)$ and $\lim_{t \to t_i^-} f(t)$ exist and finite.

A function $f$ is called piecewise continuous on $[0, +\infty)$ if it is piecewise continuous on the interval $[0, A]$ for any $A > 0$.

**Examples:**

**Definition 2.** (functions of exponential order $a$) A function $f$ is said to be of exponential order $a$ as $t \to +\infty$ if there exist real constants $M \geq 0$, $K > 0$ and $a$ such that

$$|f(t)| \leq Ke^{at}, \quad \text{for all } t \geq M.$$

**Examples:**

**Theorem 1.** (on existence of Laplace transform) If $f(t)$ is piecewise continuous on $[0, +\infty]$ and of the exponential order $a$ then the Laplace transform $F(s)$ of $f(t)$ exists for any $s > a$. Moreover, $|F(s)| \leq L/s$ for some positive constant $L$.

**Sketch of the proof:**
Main properties (regarding problems of section 6.2):

(1) Translation in $s$:
\[ \mathcal{L} \left\{ e^{\alpha t} f(t) \right\} = F(s - \alpha); \]

Proof.

\[ \mathcal{L} \left\{ e^{\alpha t} \sin \beta t \right\} = \]

\[ \mathcal{L} \left\{ e^{\alpha t} \cos \beta t \right\} = \]

(2) Laplace transform of the derivative:
\[ \mathcal{L} \left\{ f'(t) \right\} = s \mathcal{L} \{ f(t) \} - f(0) \]

More generally,
\[ \mathcal{L} \left\{ f''(t) \right\} = s^2 \mathcal{L} \{ f(t) \} - sf(0) - f'(0), \]

By induction,
\[ \mathcal{L} \left\{ f^{(n)}(t) \right\} = s^n \mathcal{L} \{ f \} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0); \]

(3) Derivative of Laplace transform:
\[ \mathcal{L} \left\{ t^n f(t) \right\} = (-1)^n F^{(n)}(s). \]

Example \[ \mathcal{L} \left\{ t^n e^{\alpha t} \right\} = \]

Theorem 2. (Existence of the inverse Laplace transform) If $f$ and $g$ are piecewise continuous functions of exponential order $a$ on $[0, +\infty)$ and they have the same Laplace transform, i.e. \( \mathcal{L} \{ f(t) \} \equiv \mathcal{L} \{ g(t) \}, \) then $f(t) = g(t)$ at all points of continuity of the functions $f$ and $g$. In particular, if $f$ and $g$ are continuous then $f \equiv g$. 