REVIEW: Power Series

DEFINITION 1. A power series about \( x = x_0 \) (or centered at \( x = x_0 \)), or just power series, is any series that can be written in the form

\[
\sum_{n=0}^{\infty} a_n (x - x_0)^n,
\]

where \( x_0 \) and \( a_n \) are numbers. The \( a_n \)'s are called the coefficients of the power series.

Absolute Convergence: The series \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) is said to converge absolutely at \( x \) if

\[
\sum_{n=0}^{\infty} |a_n| |x - x_0|^n
\]

converges.

If a series converges absolutely then it converges (But in general not vice versa).

EXAMPLE 2. The series \( \sum_{n=1}^{\infty} \frac{x^n}{n} = \lim_{m \to \infty} \) converges at \( x = -1 \), but it doesn’t converge absolutely:

\[
1 - \frac{1}{2} + \frac{1}{3} - \ldots = \ln 2
\]

but

\[
1 + \frac{1}{2} + \frac{1}{3} + \ldots
\]

is divergent.

Fact: If the series \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) converges absolutely at \( x = x_1 \) then it converges absolutely for all \( x \) such that \( |x - x_0| < |x_1 - x_0| \)

THEOREM 3. For a given power series \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) there are only 3 possibilities:

1. The series converges only for \( x = x_0 \).
2. The series converges for all \( x \).
3. There is \( R > 0 \) such that the series converges if \( |x - x_0| < R \) and diverges if \( |x - x_0| > R \). We call such \( R \) the radius of convergence.

REMARK 4. In case 1 of the theorem we say that \( R = 0 \) and in case 2 we say that \( R = \infty \)

How to find Radius of convergence: If \( a_n \neq 0 \) for any \( n \) and \( \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \) exists, then

\[
R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|^{-1}
\]
Geometric Series: \(1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \lim_{m \to \infty} \sum_{n=1}^{m} x^n = \lim_{m \to \infty} \sum_{n=1}^{m} \frac{1 - x^{m+1}}{1 - x} = \frac{1}{1 - x}\)

provided \(|x| < 1\). The series diverges if \(|x| \geq 1\). We can use also the ratio test: \(a_n = 1\) and then \(R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = 1\)

Another example: for \(\sum_{n=1}^{\infty} \frac{x^n}{n}\) we have \(a_n = \frac{1}{n}\) and thus

\[
R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n \to \infty} \frac{n + 1}{n} = 1
\]

The Taylor series for \(f(x)\) about \(x = x_0\)

Assume that \(f\) has derivatives of any order at \(x = x_0\). Then

\[
f(x) = \sum_{n=0}^{m} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + \frac{f^{(m+1)}(c)}{(m+1)!} (x - x_0)^{m+1}
\]

where \(c\) is between \(x\) and \(x_0\). The remainder converges to zero at least as fast as \((x - x_0)^{m+1}\) when \(x \to x_0\). Formally we can consider the following power series:

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n
\]

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\]

\[
\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots
\]

\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]