ON GEOMETRY OF CURVES OF FLAGS OF CONSTANT TYPE

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ABSTRACT. We develop an algebraic version of Cartan's method of equivalence or an analog of Tanaka prolongation for the (extrinsic) geometry of curves of flags of a vector space W with respect to the action of a subgroup G of $GL(W)$. Under some natural assumptions on the subgroup G and on the flags, one can pass from the filtered objects to the corresponding graded objects and describe the construction of canonical bundles of moving frames for these curves in the language of pure linear algebra. The scope of applicability of the theory includes geometry of natural classes of curves of flags with respect to reductive linear groups or their parabolic subgroups. As simplest examples, this includes the projective and affine geometry of curves. The case of classical groups is considered in more detail.

1. INTRODUCTION

Fix a vector space W over a field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Also, fix integers $0 = k_0 \leq k_1 \leq k_2 \leq \ldots \leq k_\mu = \dim W$ and let $F_{k_1,\ldots,k_{\mu-1}}(W)$ be the manifold of all flags $0 = \Lambda_0 \subseteq \Lambda_{-1} \subseteq \Lambda_{-2} \subseteq \ldots \subseteq \Lambda_{-\mu} = W$, where Λ_{-i} are k_i-dimensional linear subspaces. We allow equalities among k_i, i.e. repeated subspaces in flags, because direct sums of flags will play an important role in the sequel.

Now fix a Lie subgroup G of $GL(W)$. The group $GL(W)$ acts naturally on $F_{k_1,\ldots,k_{\mu-1}}(W)$. Assume that $\mathcal{O}$ is an orbit in $F_{k_1,\ldots,k_{\mu-1}}(W)$ with respect to the action of G. The general question is whether given two *unparamerized* curves in $\mathcal{O}$ there exists an element of G sending one curve to the other. Such two curves are said to be *G-equivalent*. We are also interested in the same question for parametrized curves of flags.

Note that particular examples of this setup include the classical projective and affine geometries of curves in $\mathbb{P}^n$ and A^n and the projective geometry of ruled surfaces. In all these cases the action of the group G is transitive on the corresponding flag varieties. Other examples we will consider in this paper include:

(1) $G = GL(W)$ and $\mathcal{O}$ is the whole flag variety;
(2) $G = Sp(W)$, if W is equipped with a symplectic form σ, $G = O(W)$, if W is equipped with a non-degenerate symmetric form Q, and $\mathcal{O}$ is the isotropic/coisotropic flag variety.

Our original motivation to study such equivalence problems comes from the new approach, so-called *symplectification procedure*, to the geometry of structures of nonholonomic nature on manifolds such as vector distributions, sub-Riemannian structure etc. This approach was proposed in [2, 3, 4, 5] and it is based on the Optimal Control Theory. It consists of the reduction of the equivalence problem for such nonholonomic geometric structures to the (extrinsic) differential geometry of curves in Lagrangian Grassmannians and, more generally, of curves of flags of isotropic and coisotropic subspaces in a linear symplectic space with respect to the action of the linear symplectic group (i.e. a particular case of item (2) above). The symplectification procedure was

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applied to the equivalence problem of vector distributions of rank 2 and 3 ([13, 14, 15]). For rank 2 distributions curves of flags appearing in this approach are curves of complete flags consisting of all osculating subspaces of the curve of one-dimensional subspaces of these flags, i.e. one arrives at the classical Wilczynski theory of non-degenerate curves in projective spaces [37]. However, the geometry of curves of isotropic/coisotropic flags appearing in the symplectification procedure for rank 3 distributions is more involved and needed the development of a new technique. In [15] we treated such curves by a brute force method that cannot be extended to the curves appearing in the theory of distributions of higher rank. The theory developed here gives a conceptual way to work with all such curves.

A general procedure for the equivalence problems under consideration was developed already by E. Cartan with his method of moving frames (see [7] and modern expositions, for example, by P. Griffiths [24] , M. Green [23], and M. Fels and P. Olver [18], [19]). In the present paper we distinguish curves of flags for which the construction of canonical bundle of moving frames with respect to the action of a given group G can be done in purely algebraic way and describe this construction in the language of pure Linear algebra.

For a different type of equivalence problems such as equivalence of filtered structures on manifolds an algebraic version of Cartan's equivalence method was developed by N. Tanaka in [35]. Instead of doing concrete normalizations, Tanaka describes the prolongation procedure for all possible normalizations in purely algebraic terms via so-called universal algebraic prolongation of the symbol of the filtered structure.

We develop a similar algebraic theory for unparametrized curves of flags, satisfying some natural assumptions. The constructions and the results of the paper can be almost verbatim generalized to embedded submanifolds of flag varieties (see subsection 4.6). It is worth to notice that an analog of Tanaka theory for curves (and submanifolds) in projective spaces and more general flag varieties was already developed in works of Y. Se-ashi [33, 34], interpreting geometrically and generalizing the classical work of Wilczynski [37] (see also [10]). However, Se-ashi treated a much more restrictive class of equivalence problems compared to our present paper: first, he considers the case $G = GL(W)$ only and second, he assumes that the algebraic prolongation of the symbol of the curve is semi-simple. The last assumption allows him to associate to a curve of flags (and, more generally, to a submanifold in a flag variety) a Cartan connection with values in the algebraic prolongation of the symbol by analogy with [36].

For the theory of curves (and, more generally, submanifolds) of flags our paper can be related to Se-ashi works [33, 34] in the same way as Tanaka paper [35] about filtered structures on manifolds with general constant symbol is related to his later work [36] about filtered structures with symbol having semisimple universal algebraic prolongation.

For unparametrized curves in Lagrangian Grassmannians the first nontrivial invariants were constructed in the earlier works of the second author with A. Agrachev [4, 6], using the notion of cross-ratio of four points in Lagrangian Grassmannians. Our constructions here give a way to construct a complete system of invariants for curves of flags in a much more general situation.

The present paper is closely related to our recent preprint [16] on geometry of curves in parabolic homogeneous space and, more generally, in a homogeneous spaces of a Lie group with a Lie algebra endowed with the fixed grading. The link between these papers is given in Remark 3.1. In [16] we work with abstract groups while in the present paper we work with their representations. We had to develop here a special language which is more adapted to the case of representations and to the forthcoming applications to the geometry of distributions.

The corresponding modification of the theory in the case of *parametrized* curves is given as well (see subsection 4.5). This modification give more conceptual point of view on constructions

of papers [40, 41, 42] on parametrized curves in Lagrangian Grassmannians and extend them and results of [4, 5, 28, 29, 30, 31, 32] to more general classes of curves.

Let us briefly describe the main constructions and the structure of the present paper. As in the Tanaka theory for filtered structures on manifolds, the main idea of our approach is to pass from the filtered objects to the corresponding graded objects. In order to make it work we need additional assumptions on the group G and on the chosen orbit $\mathcal{O} \subset F_{k_1,\ldots k_m}$ with respect to the action of G. This assumptions are discussed in section 2 (see Assumption 1 there). Shortly speaking, any flag $f_0 \in \mathcal{O}$ induces the filtration on the Lie algebra g of the Lie group G. And the compatibility of $\mathcal{O}$ with respect to the grading means that g is isomorphic (as a filtered Lie algebra) to the associated graded Lie algebra $\mathrm{gr}_{f_0} g$ so that passing to the graded objects we do not change the group in the equivalence problem. Note that $\mathrm{gr}_{f_0} g$ can be identified with a subalgebra of $\mathfrak{gl}(\mathrm{gr}_{f_0} W)$, where $\mathrm{gr}_{f_0} W$ is the graded space corresponding to the flag (the filtration) f_0. We give an explicit algorithm for constructing of all orbits compatible with respect to the grading under the assumption that G is semisimple (see Proposition 2.1 for the irreducible case) and apply it when G is a symplectic or orthogonal subgroups of $GL(W)$ (see Proposition 2.2 and Remark 2.1, respectively).

We also make additional assumptions on the curves of flags under consideration called compatibility with respect to differentiation (see Assumption 2 in section 3). Informally speaking, it means that the tangent line to a curve $\tau \mapsto \Lambda(\tau) \subset \mathcal{O}$ at the point $\Lambda(t)$ is a degree -1 element of the graded space $\mathrm{gr}_{\Lambda(t)} g \subset \mathfrak{gl}(\mathrm{gr}_{f_0} W)$. The condition of compatibility with respect to differentiation is natural through the refinement (osculation) procedure on curves of flags described in section 6. For example, starting with a curve in Grassmannian, it is natural to produce the curve of flags compatible with respect to differentiation by taking iteratively the osculation subspaces of the original curve. Then the equivalence problem for curves in Grassmannian is reduced to the geometry of curves in certain flag manifold which are compatible with respect to differentiation. In particular, in this way geometry of so-called non-degenerate curves in projective space is reduced to geometry of curves of complete flags compatible with respect to differentiation.

Further, in section 3, similarly to Tanaka theory, we define the symbol of a curve $\tau \mapsto \Lambda(\tau) \subset \mathcal{O}$ at a point with respect to the group G. For this first we identify the space $\mathrm{gr}_{\Lambda(t)} W$ with a fixed "model" graded space V such that this identification conjugates the group G with the fix subgroup $\mathcal{G}$ of $GL(V)$. Then the tangent line to the curve $\tau \mapsto \Lambda(\tau)$ at the point $\Lambda(t)$ can be identified with a line of degree -1 endomorphisms in the Lie algebra $\mathfrak{g}$ of the Lie group $\mathcal{G}$. Let $\mathcal{G}_0$ be the subgroup of $\mathcal{G}$ preserving the grading on V. Taking all possible identifications of $\mathrm{gr}_{\Lambda(t)} W$ with V as above we assign to the tangent line to the curve $\tau \mapsto \Lambda(\tau)$ at $\Lambda(t)$ the orbit of a degree -1 endomorphism from $\mathfrak{g}$ with respect to the adjoint action of $\mathcal{G}_0$. This orbit is called the *symbol of the curve* $\tau \mapsto \Lambda(\tau) \subset \mathcal{O}$ *at* $\Lambda(t)$ *with respect to the group* G.

The symbol of the curve of flags at a point is the basic invariant of the curve at this point. The main goal of the present paper is to study the equivalence problem (w.r.t. to the group G) for curves of flags with a given constant symbol. Note that the condition of constancy of symbol is often not restrictive. For example, this is the case, when G is semismple (or, more generally, reductive). Due to E.B. Vinberg ([38]) in this case the set of all possible symbols (for given group G) is finite. Therefore, the symbol of a curve of flags with respect to a semisimple (reductive) group G is constant in a neighborhood of a generic point.

The way to solve the equivalence problem under consideration is to associate canonically the bundle of moving frames to any curve of flags. The main result of the paper (Theorem 4.1) shows that in the case of curves with constant symbol the construction of such canonical bundle of moving frames can be done in purely algebraic way, namely in terms of so-called universal algebraic

prolongation of the symbol. The universal algebraic prolongation of the symbol or, more precisely, of the line of degree -1 endomorphisms, representing the symbol, is the largest graded subalgebra of $\mathfrak{g}$ such that its component corresponding to the negative degrees coincides with this chosen line of degree -1 endomorphisms. It is isomorphic to the algebra of infinitesimal symmetries of the so-called flat curve, which is the simplest (the most symmetric) curve among all curves with this symbol. In the proof of the main theorem, given in section 5, we first fix the normalization condition by choosing a complementary subspace to the image of certain coboundary operator in the space of certain 1-cochains. The construction of the bundle of canonical moving frames for any curve with given constant symbol imitates the construction of such bundle for the flat curve with this symbol.

It is important to emphasize that the number of prolongation steps and the dimension of the resulting bundle of moving frames is independent of the choice of the normalization condition but it depends on the symbol only: the number of prolongations steps is equal to the maximal degree in the grading of the universal algebraic prolongation of the symbol and the dimension of the bundle of moving frames is equal to the dimension of the universal algebraic prolongation of the symbol. The computation of the universal algebraic prolongation is an iterative process, where on each step one needs to solve a system of linear equations. Hence *even without fixing the normalization condition and starting the construction of canonical moving frames one can predict the main features of this construction using linear algebra only.*

Consequently, in order to apply our theory for equivalence of curves of flags with respect to the given group G it is important to classify all possible symbols with respect to this group and to calculate their universal algebraic prolongation. We implement these two tasks in sections 7 and 8, respectively, for the standard representation of classical groups. Essentially the classification of symbols boils down to a little bit finer classification than the classification of nilpotent endomorphisms via the Jordan Normal Form, taking into account the grading of the vector space on which the endomorphism acts and the presence of a symplectic or symmetric form. The universal algebraic prolongation in these cases can be effectively described using the theory of $\mathfrak{sl}_2$-representations.

Our results in the case of symplectic group are crucial for application of so-called symplectification procedure to geometry of vector distributions: they give much more conceptual view on our constructions in [15] for rank 3 distributions and will be used in our future work on distributions of arbitrary rank. Therefore the symplectic case is treated in detail. The case of orthogonal group is very similar to the case of symplectic group. Hence we will only sketch this case referring to the corresponding objects in the symplectic case.

Note that the set of all possible symbols of curves of flags with respect to a group G depends only on the group G as an abstract Lie group and it does not depend on a particular representation of G. Therefore the classification of the symbols in section 7 and the calculation of section 8 can be used for any representation of the classical groups.

In general the bundles of moving frames obtained in Theorem 4.1 do not have a structure of a principal bundle. They belong to a wider class of bundles that we call *quasi-principal bundles* (see Definition 4.3). Quasi-principal bundles have some features of the principal bundles when one passes to the grading. The question whether one can choose normalization conditions such that the resulting bundle will be a principal one is reduced to the question whether this normalization condition, as a complementary subspace to the image of certain coboundary operator in the space of certain 1-cochains, is invariant with respect to the natural action of the subgroup of the group of symmetries of the flat curve preserving a point. In general, such invariant normalization

condition may not exist. Some conditions for existence of the invariant normalization conditions and examples of symbols for which they do not exist are given in our recent preprint [16].

Finally, it is important to stress that we construct canonical bundles of moving frames for curves of flags with a given symbol in a unified way, i.e. without any branching in contrast to our previous construction for curves of isotropic/coisotropic subspaces of a linear symplectic space appearing in the symplectification procedure for rank 3 distributions ([15]) and also in contrast to the Fels-Olver approach [18, 19]. The latter was used by G. Mari Beffa (see, for example, [28, 29, 30, 31]) for geometry of parametrized curves with very particular symbols in Grassmannians of half-dimensional subspaces with respect to classical groups (of Lagrangian subspaces in the symplectic case and of isotropic half-dimensional subspaces in the orthogonal/conformal case). In the terminology of section 6 the first osculating space of such curves at any point is equal to the ambient vector space W. The main difference of those works from the treatment of the same curves in the present paper is that in those works not all curves with a given symbol but generic curves are considered. For example, the flat curves do not satisfy the genericity assumptions there.

2. Compatibility of the pair $(G, \mathcal{O})$ with respect to grading

First let us recall some basic notions on filtered and graded vector spaces. As before assume that $\mathcal{O}$ is an orbit in $F_{k_1,\ldots,k_{\mu-1}}(W)$ with respect to the action of the group G. A point

$$f_0 = \{0 = \Lambda_0 \subseteq \Lambda_{-1} \subseteq \Lambda_{-2} \subseteq \ldots \subseteq \Lambda_{-\mu} = W\}$$

of $\mathcal{O}$ is a decreasing filtration of W. So, it induces the decreasing filtration $\{(\mathfrak{gl}(W))_{f_0,i}\}_{i\in\mathbb{Z}}$ of $\mathfrak{gl}(W)$,

$$(2.1) \qquad (\mathfrak{gl}(W))_{f_0,i} = \{A \in \mathfrak{gl}(W) : A(\Lambda_j) \subset \Lambda_{j+i} \text{ for all j}\}, \quad (\mathfrak{gl}(W))_{f_0,i} \subset (\mathfrak{gl}(W))_{f_0,i-1}.$$

It also induces the filtration on any subspace of $\mathfrak{gl}(W)$. Further, let $\operatorname{gr}_{f_0} W$ be the graded space corresponding to the filtration f_0,

$$\operatorname{gr}_{f_0} W = \bigoplus_{i\in\mathbb{Z}} \Lambda_i/\Lambda_{i+1}$$

and let $\operatorname{gr}_{f_0} \mathfrak{gl}(W)$ be the graded space corresponding to the filtration (2.1),

$$\operatorname{gr}_{f_0} \mathfrak{gl}(W) = \bigoplus_{i\in\mathbb{Z}} (\mathfrak{gl}(W))_{f_0,i}/(\mathfrak{gl}(W))_{f_0,i+1}.$$

The space $\operatorname{gr}_{f_0} \mathfrak{gl}(W)$ can be naturally identified with the space $\mathfrak{gl}(\operatorname{gr}_{f_0} W)$. Indeed, if A_1 and A_2 from $(\mathfrak{gl}(W))_{f_0,i}$ belong to the same coset of $(\mathfrak{gl}(W))_{f_0,i}/(\mathfrak{gl}(W))_{f_0,i+1}$, i.e. $A_2 - A_1 \in (\mathfrak{gl}(W))_{f_0,i+1}$, and if w_1 and w_2 from Λ_j belong to the same coset of Λ_j/Λ_{j+1}, i.e. $w_2 - w_1 \in \Lambda_{j+1}$, then $A_1 w_1$ and $A_2 w_2$ belong to the same coset of $\Lambda_{j+i}/\Lambda_{j+i+1}$. This defines a linear map from $\operatorname{gr}_{f_0} \mathfrak{gl}(W)$ to $\mathfrak{gl}(\operatorname{gr}_{f_0} W)$. It is easy to see that this linear map is an isomorphism.

Now let $g \subset \mathfrak{gl}(W)$ be the Lie algebra of the group G. The filtration f_0 induces the filtration $\{g_{f_0,i}\}_{i\in\mathbb{Z}}$ on g, where

$$g_{f_0,i} = (\mathfrak{gl}(W))_{f_0,i} \cap g.$$

Let $\mathrm{gr}_{f_0} \, g$ be the graded space corresponding to this filtration. Note that the space $g_{f_0,i}/g_{f_0,i+1}$ is naturally embedded into the space $(\mathfrak{gl}(W))_{f_0,i}/(\mathfrak{gl}(W))_{f_0,i+1}$. Therefore, $\mathrm{gr}_{f_0} \, g$ is naturally embedded into $\mathrm{gr}_{f_0} \, \mathfrak{gl}(W)$ and, by above, $\mathrm{gr}_{f_0} \, g$ can be considered as a subspace of $\mathfrak{gl}(\mathrm{gr}_{f_0} \, W)$. It is easy to see that it is a subalgebra of $\mathfrak{gl}(\mathrm{gr}_{f_0} \, W)$.

In general, the algebra $\mathrm{gr}_{f_0} \, g$ is not isomorphic to the algebra g.

Example 2.1. Assume that $\dim W = 4m$, W is equipped with a symplectic (i.e. non-degenerate skew-symmetric) form σ, $Sp(W)$ is the subgroup of $GL(W)$ preserving the form σ, and $\mathfrak{sp}(W)$ is the corresponding Lie algebra. Assume that

$$f_0 = \{0 = \Lambda_0 \subset \Lambda_{-1} \subset \Lambda_{-2} = W\},$$

where $\dim \Lambda_{-1} = 2m$ and the restriction of σ to Λ_{-1} is non-degenerate. Let us prove that $\mathrm{gr}_{f_0} \, \mathfrak{sp}(W)$ is not isomorphic to $\mathfrak{sp}(W)$ (for more general case see Proposition 2.2 below). For this first identify the space W/Λ_{-1} with the skew-symmetric complement $\Lambda_{-1}^{\angle}$ of Λ_{-1} with respect to the form σ. Using this identification, we have that W/Λ_{-1} is equipped with the symplectic form, which is the restriction of σ to $\Lambda_{-1}^{\angle}$. Besides Λ_{-1} is equipped with the symplectic form, which is the restriction of σ to it. Consider the following natural decomposition

$$(2.2) \quad \mathfrak{gl}(\Lambda_{-1} \oplus W/\Lambda_{-1}) = \mathfrak{gl}(\Lambda_{-1}) \oplus \mathfrak{gl}(W/\Lambda_{-1}) \oplus \mathrm{Hom}(\Lambda_{-1}, W/\Lambda_{-1}) \oplus \mathrm{Hom}(W/\Lambda_{-1}, \Lambda_{-1}).$$

Then by direct computations one can show that the algebra $\mathrm{gr}_{f_0} \, \mathfrak{sp}(W)$ is isomorphic to the subalgebra of $\mathfrak{gl}(\Lambda_{-1} \oplus W/\Lambda_{-1})$ consisting of endomorphisms A such that if A is decomposed as $A = A_{11} + A_{22} + A_{12} + A_{21}$ with respect to (2.2) then $A_{11} \in \mathfrak{sp}(\Lambda_{-1})$, $A_{22} \in \mathfrak{sp}(W/\Lambda_{-1})$, and $A_{21} = 0$, where $\mathfrak{sp}(\Lambda_{-1})$ and $\mathfrak{sp}(W/\Lambda_{-1})$ are symplectic algebras of Λ_{-1} and W/Λ_{-1}, respectively. Consequently, the algebra $\mathrm{gr}_{f_0} \, \mathfrak{sp}(W)$ is not semisimple and is not isomorphic to $\mathfrak{sp}(W)$. More general class of examples is given by Proposition 2.2 below. On the contrary, if Λ_{-1} is a Lagrangian subspace, then similar argument shows that $\mathrm{gr}_{f_0} \mathfrak{sp}(W)$ is isomorphic to $\mathfrak{sp}(W)$. $\square$

In order that the passage to the graded objects will not change the group in the equivalence problem we have to impose that $\mathrm{gr}_{f_0} \, g$ and g are conjugate for some (and therefore any) $f_0 \in \mathcal{O}$. More precisely we will assume in the sequel the following

Assumption 1 (*compatibility with respect to the grading*) *For some $f_0 \in \mathcal{O}$, $f_0 = \{0 = \Lambda_0 \subseteq \Lambda_{-1} \subseteq \Lambda_{-2} \subseteq \ldots \subseteq \Lambda_{-\mu} = W\}$, there exists an isomorphism $J : \mathrm{gr}_{f_0} \, W \mapsto W$ such that*

(1) $J(\Lambda_i/\Lambda_{i+1}) \subset \Lambda_i$, $-\mu \leq i \leq -1$;
(2) *J conjugates the Lie algebras $\mathrm{gr}_{f_0} \, g$ and g i.e.*

$$(2.3) \qquad\qquad\qquad g = \{J \circ x \circ J^{-1} : x \in \mathrm{gr}_{f_0} \, g\}.$$

Note that from the transitivity of the action of G on $\mathcal{O}$ it follows that if Assumption 1 holds for some $f_0 \in \mathcal{O}$ then it holds for any other $f_0 \in \mathcal{O}$. If Assumption 1 holds we say that the pair $(G, \mathcal{O})$ is *compatible with respect to the grading*. Besides, the Lie algebra g has a grading via formula (2.3) and this grading is defined up to a conjugation.

Obviously, if $G = GL(W)$ or $SL(W)$, then G acts transitively on any flag variety $F_{k_1,\ldots,k_{\mu-1}}(W)$ and the pair $(G, F_{k_1,\ldots,k_{\mu-1}}(W))$ is compatible with respect to the grading. In general in order to construct a pair $(G, \mathcal{O})$ compatible with respect to the grading, one can start with the fixed $\mathbb{Z}$-grading on the Lie algebra g, $g = \bigoplus_{i \in Z} g_i$, and try to find a flag f_0 in W such that the algebra $\mathrm{gr}_{f_0} \, g$ is conjugate to g and the grading is preserved. Then as $\mathcal{O}$ one takes the orbit of f_0 with respect to G. In this case we say that the orbit $\mathcal{O}$ is *compatible with respect to the grading of g.*

In the case of semisimple g there is an explicit algorithm for constructing all orbits of flags compatible with respect to the grading of g. Recall that an element e of a graded Lie algebra $g = \bigoplus_{i \in Z} g_i$ is called a *grading element* if $ad_e(x) = ix$ for any $x \in g_i$. Since the map $\delta : g \to g$ sending $x \in g_i$ to ix is a derivation of g and any derivation of a semisimple Lie algebra is inner, for any graded semisimple Lie algebra there exists the unique grading element e. Moreover, this element is also semisimple as an endomorphism of W.

Proposition 2.1. *If $G \subset GL(W)$ is a semisimple Lie group acting irreducibly on W and a grading is fixed on its Lie algebra g, then there exists a unique orbit of flags compatible with respect to the grading of g.*

Proof. Assume that λ is the highest weight of the g-module W and ν is the corresponding lowest weight. Using the basic representation theory of semisimple Lie algebras, one can easily get that in the considered case the spectrum $\operatorname{spec}(e)$ of the grading element e satisfies

$$(2.4) \qquad \operatorname{spec}(e) = \{\lambda(e) - i : i \in \mathbb{Z}, 0 \leq i \leq \lambda(e) - \nu(e)\}$$

(note that $\lambda(e) - \nu(e)$ is a natural number). Therefore the natural order on the spectrum of e is defined. Let $W = \bigoplus_{j=\nu(e)-\lambda(e)-1}^{-1} W_j$ be the decomposition of W by the eigenspaces of e such that W_j is the eigenspace corresponding to the eigenvalue $\lambda(e) + j + 1$. Take the flag

$$(2.5) \qquad f_0 = \{W^j\}_{j=\nu(e)-\lambda(e)-1}^{-1} \text{ such that } W^j = \bigoplus_{i \geq j} W_j.$$

If $x \in g_i$ and $w \in W_j$, then $e(w) = (\lambda(e) + j + 1)w$ and $[e, x](w) = iw$. Therefore

$$(2.6) \qquad e \circ x(w) = [e, x](w) + x \circ e(w) = (\lambda(e) + (j + i) + 1)x(w),$$

i.e. $x(w) \in W_{j+i}$. This implies that the map $J : \operatorname{gr}_{f_0} W \to W$, which sends an element of W^j/W^{j+1} to its representative in W_j, conjugates $\operatorname{gr}_{f_0} g$ and g. Therefore the orbit O_{f_0} of f_0 with respect to G is compatible with respect to the grading of g.

Now let us briefly sketch the proof of uniqueness, which is based on the irreducibility assumption. Since the grading element e belongs to g_0, an orbit compatible with respect to the grading of g must contain a flag $\{\widetilde{W}^j\}_{j=-\mu}^{-1}$ such that each subspace $\widetilde{W}_j$ is an invariant subspace of e. First one proves that $W^{-1} \subset \widetilde{W}^{-1}$. Assuming the converse, it is not hard to show that the space $g.\widetilde{W}^{-1}$ is a proper subspace of W, because it does not contain the nonempty set $W^{-1}\backslash\widetilde{W}^{-1}$. This contradicts the irreducibility assumption. Further, if W^{-1} is a proper subspace of $\widetilde{W}^{-1}$, then in a similar way one can prove that the space $g.W^{-1}$ does not contain the nonempty set $\widetilde{W}^{-1}\backslash W^{-1}$ which again contradicts the irreducibility assumption. Hence, $\widetilde{W}^{-1} = W^{-1}$. In the same manner one can prove that $\widetilde{W}^j = W^j$ for any $-\mu \leq j \leq -2$. $\qquad \square$

Note that Proposition 2.1 is also true if G is reductive. The proof is the same. The only difference is that the grading element is not unique: it is defined modulo the center of g, but this fact does not affect the proof.

We are especially interested in the case, when W is an even dimensional vector spaces equipped with a symplectic form σ and $G = Sp(W)$ or $CSp(W)$, where $Sp(W)$ is the corresponding symplectic group and $CSp(W)$ is the so-called conformal symplectic group, i.e. the group of all transformation preserving the symplectic form σ up to a multiplication by a nonzero constant. Denote by $\mathfrak{sp}(W)$ and $\mathfrak{csp}(W)$ the corresponding Lie algrebras. Recall that a subspace L of W

is called *isotropic* with respect to the symplectic form σ if the restriction of the form σ on L is identically equal to zero or, equivalently, L is contained in its skew-symmetric complement $L^{\angle}$ (with respect to σ), while a subspace L of W is called *coisotropic* with respect to the symplectic form σ if L contains $L^{\angle}$.

Definition 2.1. *We say that* a flag $f_0 = \{0 = \Lambda_0 \subseteq \Lambda_{-1} \subseteq \Lambda_{-2} \subseteq \ldots \subseteq \Lambda_{-\mu} = W\}$ in W is symplectic *if* $(\Lambda_{-i})^{\angle} = \Lambda_{i-\mu}$ *for any* $0 \leq i \leq \mu$.

Obviously, the flag f_0 is symplectic, if and only if the following three conditions hold: any subspace in the flag f_0 is either isotropic or coisotropic with respect to the symplectic form σ; a subspace belongs to the flag f_0 together with its skew-symmetric complement; the number of appearances of a subspace in the flag f_0 is equal to the number of appearances of its skew-symmetric complement in f_0.

Proposition 2.2. *An orbit $\mathcal{O}$ in a flag manifold is compatible with respect to some grading on $g = \mathfrak{sp}(W)$ or $\mathfrak{csp}(W)$ if and only if some (and therefore any) flag $f_0 \in \mathcal{O}$ is symplectic.*

Proof. Let λ be the highest weight of the considered standard representation of $\mathfrak{sp}(W)$. Fix the grading on g and let e be the grading element. Since $\mathfrak{csp}(W) = \mathfrak{sp}(W) \oplus \mathbb{K}$, in the case $g = \mathfrak{csp}(W)$ we can always choose $e \in \mathfrak{sp}(W)$. By above $\mathcal{O}$ is the orbit of the flag f_0 defined by (2.5). Since $e \in \mathfrak{sp}(W)$, we have that $\sigma(ew_1, w_2) + \sigma(w_1, ew_2) = 0$ for any w_1 and w_2 in W. This implies that if $w_1 \in W_{j_1}$, $w_2 \in W_{j_2}$, and $\lambda(e) + j_1 + 1 \neq -(\lambda(e) + j_2 + 1)$ then $\sigma(w_1, w_2) = 0$, where W_j is the eigenspace of e corresponding to the eigenvalue $\lambda(e) + j + 1$. From this and the fact that the form σ is non-degenerate it follows that the spectrum of e is symmetric with respect to 0 (for more general statement see Remark 2.2 below) and σ defines non-degenerate pairing between W_{j_1} and W_{j_2} with $\lambda(e) + j_1 + 1 = -(\lambda(e) + j_2 + 1)$. Consequently, the subspaces W^j with $\lambda(e) + j + 1 > 0$ are isotropic and $W^{j_2} = (W^{j_1})^{\angle}$ for $\lambda(e) + j_1 + 1 = -(\lambda(e) + j_2 + 1)$. Thus, the flag f_0 is symplectic. $\square$

Remark 2.1. Assume that W is a vector space equipped with a non-degenerate symmetric form Q, and $G = O(W)$ or $CO(W)$, the orthogonal or conformal groups. The notion of isotropic and coisotropic subspaces of W with respect to the form Q are defined similarly to the symplectic case, using orthogonal complements instead of skew-symmetric ones. Then by complete analogy with Proposition 2.2 the orbits of flags compatible with some grading of $g = \mathfrak{so}(W)$ or $\mathfrak{cso}(W)$ consist of flags $f_0 = \{0 = \Lambda_0 \subseteq \Lambda_{-1} \subseteq \Lambda_{-2} \subseteq \ldots \subseteq \Lambda_{-\mu} = W\}$ such that $(\Lambda_{-i})^{\perp} = \Lambda_{i-\mu}$ for any $0 \leq i \leq \mu$, where $L^{\perp}$ denotes the orthogonal complement of L with respect to Q. We will call such flags *orthogonal*. A flag f_0 is orthogonal if and only if the following tree conditions hold: any Λ_i is either isotropic or coisotropic subspaces with respect to Q; a subspace belongs to the flag f_0 together with its orthogonal complement with respect to the form Q; the number of appearances of a subspace in the flag f_0 is equal to the number of appearances of its orthogonal complement in f_0. In particular, if $\mathbb{K} = \mathbb{R}$ and the form Q is sign definite then there are no orbits of flags compatible with the grading except the trivial one $0 \subset W$. $\square$

In the case of a general (not necessarily irreducible) representation of a semisimple (a reductive) Lie group G, the flags compatible with the grading can be constructed by the following algorithm:

(1) take flags as in Proposition 2.1 in each irreducible component;
(2) shift degrees of subspace in each of these flags by arbitrary nonpositive numbers with the only restriction that for at least one irreducible component there is no shift of degrees, i.e. the minimal nontrivial subspace in the flag sitting in this component has degree -1.

Consider a flag which is a direct sum of the flags constructed in each irreducible component (with shifted degrees as above): the degree i subspace of this flag is equal to the direct sum of degree i subspaces of flags in each irreducible component. Obviously, the orbits of such flags are compatible with respect to some grading of g. Moreover, they are the only orbits satisfying this property. The above restriction on the shifts of degrees was done in order that the resulting flag will satisfy our convention that the minimal nontrivial subspace in it has degree -1.

Remark 2.2. Note that the spectrum of the grading element is symmetric with respect to 0 for any representation of the following simple Lie algebras A_1, B_ℓ, C_ℓ, D_ℓ for even ℓ, E_7, E_8, F_4, and G_2 (in the proof of Proposition 2.2 it was shown for the standard representation of the symplectic Lie algebras C_ℓ only). It follows from the fact that among all simple Lie algebras these are the only algebras for which the map -1 belongs to the Weyl group of their root system (see, for example,[25, p.71, Exercise 5]). Thus the highest and the lowest weights λ and ν of any irreducible representation of these algebras satisfy $\nu = -\lambda$, which togehter with formula (2.4) implies the desired statement about the spectrum of the grading element. $\square$

As an example of non-reductive group G consider the case of an *affine subgroup* $\mathrm{Aff}(W)$ of $GL(W)$ which consists of all elements of $GL(W)$ that preserve a fixed affine hyperplane $\mathcal{A}$ of W. Obviously, the restriction of an element of $\mathrm{Aff}(W)$ to the affine space $\mathcal{A}$ is an affine transformation of $\mathcal{A}$. We define an affine flag in an affine space $\mathcal{A}$ as a set $\{\mathcal{A}_i\}_{i=-\mu}^{-1}$ of nested affine subspaces of $\mathcal{A}$, $\mathcal{A}_i \subset \mathcal{A}_{i-1}$. An affine flag is called complete if it contains affine subspaces of all possible dimensions. A flag $\{\Lambda_i\}_{i=-\mu}^{-1}$ in W with $\Lambda_{-1} \cap \mathcal{A} \neq \emptyset$ defines the affine flag $\{\Lambda_i \cap \mathcal{A}\}_{i=-\mu}^{-1}$ in $\mathcal{A}$. So, the equivalence problem for curves of flags with respect to the affine group $\mathrm{Aff}(W)$ can be reformulated as the equivalence problem for curves of affine flags with respect to the group of affine transformations of $\mathcal{A}$. As a particular case we have the classical equivalence problem for curves in an affine space. In order to use our theory for non-degenerate curves in an affine space $\mathcal{A}$, i.e. curves which do not lie in any proper affine subspace of $\mathcal{A}$, one has first to make the refinement (osculation) procedure of section 6 below to reduce the original equivalence problem to the equivalence problem of curves of complete affine flags. Finally, it is not hard to show that for any orbit $\mathcal{O}$ of flags with respect to $\mathrm{Aff}(W)$ the pair $(\mathrm{Aff}(W), \mathcal{O})$ is compatible with respect to the grading.

Remark 2.3. The treatment of the reductive case suggests a way for construction of orbits compatible with respect to the grading in the case of arbitrary (not necessary reductive) graded subalgebra $g \subset \mathfrak{gl}(W)$. Assume that a grading element e exists and also that e, as an endomorphism of W, is semisimple. Now assume that the spectrum of e is a disjoint union of the sets $\{A_j\}_{j\in\mathbb{Z}}$ such that if λ belongs to A_j and $\lambda + i$ is an eigenvalue of e then $\lambda + i \in A_{j+i}$. As a grading of W take the splitting such that the j subspace of this splitting is the the sum of the eigenspaces of elements of A_j and as a flag f_0 take the corresponding flag as in (2.5). Then by the same arguments as in the proof of Proposition 2.1 we get that the orbit of f_0 is compatible with respect to the grading of g.

3. Compatibility with respect to differentiation and symbols of curves of flags

After clarifying what kind of orbits in flag varieties will be considered, let us clarify what kind of curves in these orbits will be studied. Let

$$(3.1) \qquad t \mapsto \Lambda(t) = \{0 = \Lambda_0(t) \subseteq \Lambda_{-1}(t) \subseteq \Lambda_{-2}(t) \subseteq \ldots \subseteq \Lambda_{-\mu}(t) = W\}$$

be a smooth curve in $\mathcal{O}$. We are interested in unparametrized curves here. The constructions below do not depend on the parametrization chosen in (3.1).

Recall that for a given parametrization of the curve (3.1) the velocity $\frac{d}{dt}\Lambda_i(t)$ at t of the curve $\tau \mapsto \Lambda_i(\tau)$ can be naturally identified with an element of $\mathrm{Hom}\big(\Lambda_i(t), W/\Lambda_i(t)\big)$. Namely, given $l \in \Lambda_i(t)$ take a smooth curve of vector $\ell(\tau)$ satisfying the following two properties:

  (1) $\ell(t) = l$,
  (2) $\ell(\tau) \in \Lambda_i(\tau)$ for any τ close to t.

Note that the coset of $\ell'(t)$ in $W/\Lambda_i(t)$ is independent of the choice of the curve ℓ satisfying the properties (1) and (2) above. Then to $\frac{d}{dt}\Lambda_i(t)$ we assign the element of $\mathrm{Hom}\big(\Lambda_i(t), W/\Lambda_i(t)\big)$ which sends $l \in \Lambda_i(t)$ to the coset of $\ell'(t)$ in $W/\Lambda_i(t)$, where the curve ℓ satisfies properties (1) and (2) above. It defines a linear map from the tangent space at $\Lambda_i(t)$ to the Grassmannian of k_{-i}-dimensional subspace of W to the space $\mathrm{Hom}\big(\Lambda_i(t), W/\Lambda_i(t)\big)$. It is easy to show that this map is an isomorphism and therefore it defines the required identification.

Assumption 2 (*compatibility with respect to differentiation*) *We assume that for some (and therefore any) parametrization of the curve* (3.1) *the velocity* $\frac{d}{dt}\Lambda_i(t)$ *satisfies*

$$\frac{d}{dt}\Lambda_i(t) \in \mathrm{Hom}\big(\Lambda_i(t), \Lambda_{i-1}(t)/\Lambda_i(t)\big), \quad \forall\, -\mu + 1 \leq i \leq -1 \ \text{and}\ t.$$

In this case we say that the curve (3.1) is *compatible with respect to differentiation*. Equivalently, a curve (3.1) is compatible with respect to differentiation if for every i, $-\mu + 1 \leq i \leq -1$, the following holds: if $\ell(t)$ is a smooth curve of vectors such that $\ell(t) \in \Lambda_i(t)$ for any t, then $\ell'(t) \in \Lambda_{i-1}(t)$ for any t. The condition of compatibility with respect to differentiation is natural through the refinement procedure on curves of flags described in section 6 below.

Under Assumption 2, $\frac{d}{dt}\Lambda_i(t)$ factors through a map δ_t from $\Lambda_i(t)/\Lambda_{i+1}(t)$ to $\Lambda_{i-1}(t)/\Lambda_i(t)$. In other words, the map $\delta_t \in \mathfrak{gl}\big(\mathrm{gr}_{\Lambda(t)} W\big)$ of degree -1 is well defined up to a multiplication by a nonzero constant (recall that the reparametrization is allowed). Besides, since the curve (3.1) belongs to the orbit of G then

$$(3.2) \qquad\qquad\qquad\qquad \delta_t \in \Big(\mathrm{gr}_{\Lambda(t)}\, g\Big)_{-1}.$$

Now we define the symbol of the curve of flags $\tau \mapsto \Lambda(\tau)$ at the point $\Lambda(t)$ with respect to the group G. Shortly speaking, the symbol is a line of degree -1 endomorphisms in the Lie algebra $\mathfrak{g}$ of the Lie group $\mathcal{G}$, defined up to a conjugation, which can be naturally identified with the tangent line to the curve $\tau \mapsto \Lambda(\tau)$ at the point $\Lambda(t)$.

For a precise definition let us start with some notations. Fix $f_0 \in \mathcal{O}$ such that $f_0 = \{0 = \Lambda_0 \subset \Lambda_{-1} \subset \Lambda_{-2} \subset \ldots \subset \Lambda_{-\mu} = W\}$ and denote $V = \mathrm{gr}_{f_0} W$, $V_i = \Lambda_i/\Lambda_{i+1}$. The grading $V = \bigoplus\limits_{i=-\mu}^{-1} V_i$ defines also the natural filtration

$$(3.3) \qquad\qquad 0 \subset V^{-1} \subset V^{-2} \subset \ldots \subset V^{-\mu} = V, \quad V^i = \bigoplus_{j=i}^{-1} V_j \quad -\mu \leq i \leq -1.$$

Further, fix an isomorphism $J : V \to W$, satisfying conditions of Assumption 1. Let $\mathcal{G}$ be the subgroup of $GL(V)$ such that $\mathcal{G} = \{J^{-1} \circ A \circ J : A \in G\}$ and $\mathfrak{g}$ is its Lie algebra. Further, let

$\mathfrak{gl}(V)_k$ be the space of endomorphisms of V of degree k, $\mathfrak{gl}(V)_k = \bigoplus\limits_{i=-\mu}^{-1} \mathrm{Hom}(V_i, V_{i+k})$, and

$$(3.4) \qquad\qquad\qquad \mathfrak{g}_k = \mathfrak{g} \cap \mathfrak{gl}(V)_k.$$

By Assumption 1 this defines the grading of the Lie algebra $\mathfrak{g}$: $\mathfrak{g} = \bigoplus\limits_{k \in Z} \mathfrak{g}_k$. We define a "big" bundle $\widehat{P}$ over the orbit $\mathcal{O}$ with the fiber $\widehat{P}_\Lambda$ over a point $\widetilde{\Lambda} = \{\widetilde{\Lambda}_i\}_{i=-\mu}^{-1}$ consisting of all isomorphisms $A : V \to W$ such that

(1) A preserves the filtrations (3.3) and $\widetilde{\Lambda} = \{\widetilde{\Lambda}_i\}_{i=-\mu}^{-1}$, i.e. $A(V^i) = \widetilde{\Lambda}_i$ for any $-\mu + 1 \le i \le -1$;

(2) A conjugates the Lie groups $\mathcal{G}$ and G i.e. $G = \{A \circ X \circ A^{-1} : X \in \mathcal{G}\}$; .

(3) $A \circ J^{-1} \in G$.

Further, let $\mathcal{G}_+$ be the subgroup of $\mathcal{G}$ consisting of all elements of $\mathcal{G}$ preserving the filtration (3.3). Obviously, the corresponding subalgebra $\mathfrak{g}_+$ satisfies $\mathfrak{g}_+ = \bigoplus\limits_{k \ge 0} \mathfrak{g}_k$, where $\mathfrak{g}_k$ are as in (3.4).

It is easy to see that the bundle $\widehat{P}$ is a principal $\mathcal{G}_+$-bundle over the orbit $\mathcal{O}$.

Remark 3.1. *(Homogeneous spaces formulation)* Note that $\widehat{P}$ can be identified with G via the map $A \mapsto A \circ J^{-1}$ and with $\mathcal{G}$ via the map $A \mapsto A^{-1}J$, $A \in \widehat{P}$. Besides, under the latter identification $\mathcal{O} \cong \mathcal{G}/\mathcal{G}_+$ and the fibers of $\widehat{P}$ are exactly the left cosets of $\mathcal{G}_+$ in $\mathcal{G}$. The homogeneous space $\mathcal{O} \cong \mathcal{G}/\mathcal{G}_+$ is equipped with the natural $\mathcal{G}$-invariant vector distribution D which is equal at the "origin" of $\mathcal{G}/\mathcal{G}_+$ (i.e. at the coset of identity of $\mathcal{G}$) to the equivalent classes of elements of degree -1 of $\mathfrak{g}$ under the identification of the tangent space to $\mathcal{G}/\mathcal{G}_+$ at the "origin" with $\mathfrak{g}/\mathfrak{g}_+$. Then a curve in $\mathcal{O}$ is compatible with respect to differentiation if and only if it is an integral curve of the distribution D. So, *the equivalence problem for curves of flags compatible with respect to differentiation can be reformulated as the equivalence problem for integral curves of the natural $\mathcal{G}$-invariant distribution in the homogeneous space $\mathcal{G}/\mathcal{G}_+$.* This point of view, restricted to the parabolic homogeneous spaces, is considered in our recent preprint [16] □

Further, the group $\mathcal{G}_+$ acts naturally on $\mathfrak{g}_{-1}$ as follows: $A \in \mathcal{G}_+$ sends $x \in \mathfrak{g}_{-1}$ to the degree -1 component of $(\mathrm{Ad}\,A)\,x$. It induces the action on the projectivization $\mathbb{P}\mathfrak{g}_{-1}$ in the obvious way.

Definition 3.2. The set $\mathfrak{M}_t = \{\mathbb{K}\big(A^{-1} \circ \delta_t \circ A\big)_{-1} : A \in \widehat{P}_{\Lambda(t)}\}$, where $\big(A^{-1} \circ \delta_t \circ A\big)_{-1}$ denotes the degree -1 component of $A^{-1} \circ \delta_t \circ A$, is an orbit in $\mathbb{P}\mathfrak{g}_{-1}$ with respect to aforementioned action of $\mathcal{G}_+$ This orbit is called the *symbol of the curve* (3.1) *at the point* $\{\Lambda_i(t)\}_{i=-1}^{-\mu}$ *with respect to* G.

Remark 3.2. Note that by definition the set of all possible symbols of curves of flags with respect to a group G depends only on the group G as an abstract Lie group. In other words, it does not depend on a particular embedding of G to $GL(W)$ for some W or, equivalently, on a particular faithful representation of G. □

In the sequel we will consider curves of flags with the constant symbol $\mathfrak{M}$, i.e. $\mathfrak{M}_t = \mathfrak{M}$ for any t. We also say that such curves of flags are *of constant type* $\mathfrak{M}$. If G (and therefore $\mathcal{G}$) is semisimple and $\mathcal{G}_0$ is the connected subgroup of $\mathcal{G}$ with subalgebra $\mathfrak{g}_0$, then due to E.B. Vinberg [38] the set of orbits with respect to the adjoint action of G_0 on $\mathbb{P}\mathfrak{g}_{-1}$ is finite. Note that if e is the grading element of $\mathfrak{g}$ and $\widetilde{\mathcal{G}}_0$ is the stabilizer of e with respect to the adjoint action of $\mathcal{G}$, then the orbits of $\mathfrak{g}_{-1}$ with respect to the natural action of $\mathcal{G}_+$ and the adjoint action of $\widetilde{\mathcal{G}}_0$ coincide. Besides $\mathcal{G}_0$ is just the connected component of the identity in $\widetilde{\mathcal{G}}_0$. Therefore *in the case when G*

is semisimple the set of all possible symbols is finite and the condition of constancy of the symbol holds in a neighborhood of a generic point of a curve. The same conclusions can be done if G is reductive.

Any curve of flags that is G-equivalent to the curve $t \mapsto \{J \circ e^{t\delta}V^i\}_{i=0,-1\ldots,-\mu}$ for some δ such that $\mathbb{K}\delta \in \mathfrak{M}$ is called the *flat curve with constant symbol* $\mathfrak{M}$ (here $V_0 = 0$). The flat curve is an essence the simplest curve among all curves with a given symbol.

Remark 3.3. If the group G coincides with its normalizer in $GL(W)$, then for all constructions above it is not necessary to fix a map $J : V \mapsto W$ and the condition (3) in the definition of the bundle $\widehat{P}$ can be omitted.

Example 3.1. Assume that $\dim W = n + 1$ and consider a curve of complete flags

$$t \mapsto \{0 \subset \Lambda_{-1}(t) \subset \Lambda_{-2}(t) \subset \ldots \Lambda_{-(n+1)}(t) = W\},$$

compatible with respect to differentiation. A complete flag means that $\dim \Lambda_{-i} = i$. Assume also that for any $-n < i < -1$ the velocity $\frac{d}{dt}\Lambda_i(t)$, as an element of $\mathrm{Hom}(\Lambda_i(t), \Lambda_{i-1}(t)/\Lambda_i(t))$, is onto $\Lambda_{i-1}(t)/\Lambda_i(t)$. The symbol of such curve of complete flags (with respect to $GL(W)$) is a line of degree -1 endomorphisms of the corresponding graded spaces, generated by an endomorphism which has the matrix equal to a Jordan nilpotent block in some basis. The flat curve of the maximal refinement is the curve of osculating subspaces of a rational normal curve in the projective space $\mathbb{P}W$. Recall that a rational normal curve in $\mathbb{P}W$ is a curve represented as $t \mapsto [1 : t : t^2 : \ldots t^n]$ in some homogeneous coordinates. $\square$

As in the Tanaka theory for filtered structures on manifolds, we want to investigate the original equivalence problem via the passage to the graded objects, that is to imitate the construction of the bundle of canonical moving frames for any curve with a given constant symbol $\mathfrak{M}$ via the construction of such bundle for the flat curve with symbol $\mathfrak{M}$. The latter can be done in purely algebraic way via the notion of the universal algebraic prolongation of the symbol which is introduced in the next section.

4. Algebraic prolongation of the symbol and the main result

From now on we consider a curve of flags (3.1) with constant symbol $\mathfrak{M}$ and we fix a line $\mathfrak{m}$ in $\mathfrak{g}_{-1}$ representing the orbit $\mathfrak{M}$. Often the line $\mathfrak{m}$ itself will be called the symbol of the curve (3.1) as well and we will shortly say that the curve (3.1) has constant symbol $\mathfrak{m}$ instead of constant symbol with the representative $\mathfrak{m}$.

4.1. **Algebraic prolongation of symbol.** Set $\mathfrak{u}_{-1} = \mathfrak{m}$ and define by induction in k

$$(4.1) \qquad\qquad \mathfrak{u}_k := \{X \in \mathfrak{g}_k : [X, \delta] \in \mathfrak{u}_{k-1}, \delta \in \mathfrak{m}\}, \quad k \geq 0.$$

The space $\mathfrak{u}_k$ is called the *kth algebraic prolongation of the line* $\mathfrak{m}$. Then by construction $\mathfrak{u}(\mathfrak{m}) = \bigoplus_{k \geq -1} \mathfrak{u}_k$ is a graded subalgebra of $\mathfrak{g}$. It can be shown that it is *the largest graded subalgebra of $\mathfrak{g}$ such that its component corresponding to the negative degrees coincides with* $\mathfrak{m}$. The algebra $\mathfrak{u}(\mathfrak{m})$ is called the *universal algebraic prolongation of the line $\mathfrak{m}$ (of the symbol $\mathfrak{M}$)*. Obviously, $\mathfrak{gl}(V)_k = 0$ for all $k \geq \mu$. So $\mathfrak{u}_k = 0$ for $k \geq \mu$.

The algebra $\mathfrak{u}(\mathfrak{m})$ has a very natural geometric meaning. Namely, let Γ be the curve $t \mapsto \{e^{t\delta}V^i\}_{i=0,-1,\ldots,-\mu}$ in the corresponding flag manifold $F_{k_1,\ldots,k_{\mu-1}}(V)$ for some $\delta \in \mathfrak{m}$. The group $GL(V)$ acts naturally on this flag manifold, and thus, we can identify each element $X \in \mathfrak{gl}(V)$

with a vector field on $F_{k_1,\ldots,k_{\mu-1}}(V)$. We define a symmetry algebra of Γ as a set of all elements $X \in \mathfrak{g}$, such that the corresponding vector field on $F_{k_1,\ldots,k_{\mu-1}}(V)$ is tangent to Γ.

As it is shown in [11], the symmetry algebra of Γ is the *largest subalgebra of* $\mathfrak{g}$ *that contains* $\mathfrak{m}$ *and lies in* $\mathfrak{m} + \sum_{i \geq 0} \mathfrak{g}_i$. It is easy to see from (4.1) that $\mathfrak{u}$ satisfies this property by construction. Thus, we see that $\mathfrak{u}(\mathfrak{m})$ is exactly a symmetry algebra of the curve Γ. Besides, a flat curve with symbol $\mathfrak{m}$ is G-equivalent to the curve $J(\Gamma)$. Therefore the algebra $\mathfrak{u}(\mathfrak{m})$ is conjugate to the symmetry algebra of a flat curve with the constant symbol $\mathfrak{m}$.

4.2. **Zero degree normalization.** Let $\widehat{P}|_{\Lambda(\cdot)}$ be the union of all fibers of the bundle $\widehat{P}$ over our curve (3.1), where $\widehat{P}$ is the bundle over the orbit $\mathcal{O}$ defined in section 3. Our goal is to assign to the curve $\Lambda(\cdot)$ in a canonical way a fiber subbundle of $\widehat{P}|_{\Lambda(\cdot)}$ endowed with a canonical Ehresmann connection, i.e. with a rank 1 distribution transversal to its fibers. We shall construct this bundle through the iterative construction of decreasing sequence of fiber subbundles of $\widehat{P}|_{\Lambda(\cdot)}$. Define a map

$$(4.2) \qquad \Pi_k^t : \bigoplus_{i=-\mu}^{-1} \mathrm{Hom}\big(V_i, \Lambda_i(t)\big) \to \bigoplus_{i=-\mu}^{-1} \mathrm{Hom}\big(V_i, \Lambda_i(t)/\Lambda_{i+k+1}(t)\big)$$

as follows:

$$\Pi_k^t(A)|_{V_i} \equiv A|_{V_i} \mod \Lambda_{i+k+1}(t), \quad \forall -\mu \leq i \leq -1$$

(here we assume that $\Lambda_i = 0$ for $i \geq 0$). Speaking informally, the map $\Pi_k^t(A)$ is defined by applying the appropriate canonical projecons to the values of the map A. Let $\widehat{P}_0$ be the subbundle of $\widehat{P}|_{\Lambda(\cdot)}$ with the fiber $\widehat{P}_0(t)$ over the point $\{\Lambda_i(t)\}_{i=-\mu}^{-1}$ consisting of all $A \in \widehat{P}_{\Lambda(t)}$ such that

$$\mathfrak{m} = \mathbb{K}\big(A^{-1} \circ \delta_t \circ A\big)_{-1}.$$

We also denote by P_0 the bundle over our curve with the fiber $P_0(t)$ over the point $\{\Lambda_i(t)\}_{i=-\mu}^{-1}$ equal to the image of the corresponding fiber $\widehat{P}_0(t)$ of the bundle $\widehat{P}_0$ under the map Π_0^t. By constructions P_0 is a principal U_0-bundle, where U_0 is a subgroup of $\mathcal{G}$ consisting of all degree 0 elements B of $\mathcal{G}$ such that $\mathrm{Ad}B(\mathfrak{m}) = \mathfrak{m}$. The Lie algebra of U_0 is equal to $\mathfrak{u}_0$ defined by (4.1).

4.3. **Quasi-principal subbundle of** $\widehat{P}_0$. Take a fiber subbundle P of $\widehat{P}_0$ which is not necessary a principal subbundle of $\widehat{P}_0$. Let $P(t)$ be the fiber of P over the point $\{\Lambda_i(t)\}_{i=-1}^{-\mu}$. Take $\psi \in P(t)$. The tangent space $T_\psi\big(P(t)\big)$ to the fiber $P(t)$ at a point ψ can be identified with a subspace of $\mathfrak{g}_+$. Indeed, define the following $\mathfrak{g}_+$-valued 1-form ω on P: to any vector X belonging to $T_\psi\big(P(t)\big)$ we assign an element $\omega(\psi)(X)$ of $\mathfrak{g}_+$ as follows: if $s \to \psi(s)$ is a smooth curve in $P(t)$ such that $\psi(0) = \psi$ and $\psi'(0) = X$ then let

$$(4.3) \qquad \omega(\psi)(X) = \psi^{-1} \circ X,$$

where in the last formula by ψ we mean the isomorphism between V and W. Note that the linear map $\omega(\psi) : T_\psi\big(P(t)\big) \mapsto \mathfrak{g}_+$ is injective. Set

$$(4.4) \qquad L_\psi := \omega\big(T_\psi\big(P(t)\big)\big).$$

If P is a principal bundle over our curve, which is a reduction of the bundle $\widehat{P}_0$, then the space L_ψ is independent of ψ and equal to the Lie algebra of the structure group of the bundle P. For our purposes here we need to consider more general class of fiber subbundles of $\widehat{P}_0$. To define this class recall that the filtration given by the subspaces V^i, $-\mu \leq i \leq -1$, defined by (3.3), induces a

natural filtration on $\mathfrak{gl}(V)$ and, therefore, on the subspace L_ψ. The corresponding graded subspace L_ψ is called a *symbol of the bundle P at a point ψ*. Under the natural identification of spaces $\operatorname{gr}\mathfrak{gl}(V)$ with $\mathfrak{gl}(\operatorname{gr} V)$, described in the beginning of section 2, one has that $\operatorname{gr} L_\psi$ is a subspace of $\mathfrak{gl}(\operatorname{gr} V)$. Besides V is a graded space by definition, i.e $V \sim \operatorname{gr} V$. Therefore, $\operatorname{gr} L_\psi$ is a subspace of $\mathfrak{gl}(V)$.

Definition 4.3. *We say that the bundle P has a* constant symbol $\mathfrak{s}$ *if its symbols at different points coincide with $\mathfrak{s}$. In this case we call P the* quasi-principal subbundle of the bundle $\widehat{P}_0$ with symbol $\mathfrak{s}$.

4.4. **Structure function associated with Ehresmann connection.** Further, assume that a fiber subbundle P of $\widehat{P}_0$ is endowed with an Ehresmann connection, i.e. with a rank 1 distribution $\mathcal{H}$ transversal to its fibers. A parametrized smooth curve $t \mapsto \psi(t)$ in P is called a *moving frame of the pair $(P, \mathcal{H})$* if the following two conditions holds:

   (1) the curve $t \mapsto \psi(t)$ is tangent to the distribution $\mathcal{H}$ at any point;
   (2) there exists $\delta \in \mathfrak{m}$ such that $\left(\psi^{-1} \circ \psi'(t)\right)_{-1} = \delta$ for any t.

A pair $(P, \mathcal{H})$ will be called a *bundle of moving frames*.

   Let $C^1 = \operatorname{Hom}(\mathfrak{m}, \mathfrak{g})$. Also, given $A \in \mathfrak{gl}(V)$ denote by $(A)_k$ the component of degree k of A (w.r.t. the splitting $\mathfrak{gl}(V) = \bigoplus_{k \in \mathbb{Z}} \mathfrak{gl}(V)_k$). Then to any bundle P with the fixed Ehresmann connection $\mathcal{H}$ one can assign the function $c : P \to C^1$ as follows: Given $\psi \in P(t)$ and $\delta \in \mathfrak{m}$ let $\psi(\tau)$ be a curve in P tangent to the Ehresmann connection $\mathcal{H}$ defined for $\tau \in (t - \varepsilon, t + \varepsilon)$ for some $\varepsilon > 0$ such that $\psi(t) = \psi$ and $\left(\psi^{-1} \circ \psi'(t)\right)_{-1} = \delta$. Then set

$$(4.5) \qquad\qquad c(\psi)(\delta) := \psi^{-1}(t) \circ \psi'(t).$$

Note that the right-hand side of (4.5) does not depend on the choice of the curve $\psi(\tau)$ with the aforementioned properties. The function c is called the *structure function of the bundle of moving frames $(P, \mathcal{H})$*.

Remark 4.1. Under the identification of the bundle $\widehat{P}$ with the Lie group $\mathcal{G}$ given in Remark 3.1 one can describe the structure function $c(\psi)$ in terms of the *left-invariant Maurer-Cartan form Ω* of $\mathcal{G}$ by the following formula:

$$(4.6) \qquad\qquad c(\psi)\left(\left(\Omega(X)\right)_{-1}\right) = \Omega(\psi)(X) \quad \forall \psi \in P \text{ and } X \in \mathcal{H}(\psi),$$

where $\left(\Omega(X)\right)_{-1}$ is the degree -1 component of $\Omega(X)$ with respect to the grading on $\mathfrak{g}$. Note also that the $\mathfrak{g}_+$-valued 1-form ω defined by (4.3) is nothing but the restriction of the left-invariant Maurer-Cartan form to the fibers of the bundle P. $\square$

   Note that

$$(4.7) \qquad\qquad C^1 = \bigoplus_{k \in \mathbb{Z}} C_k^1,$$

where $C_k^1 = \operatorname{Hom}(\mathfrak{m}, \mathfrak{g}_{k-1})$. Let $C_+^1 := \bigoplus_{k>0} C_k^1$, c_k be the kth component of c w.r.t. the splitting (4.7), and $c_+ = \sum_{k>0} c_k$. We say that c_+ is the *positive part of the structure function of c*. Note that by the constructions $c_k = 0$ for $k < 0$ and $c_0(\delta) = \delta$ for every $\delta \in \mathfrak{m}$.

   Further, let $\partial : \mathfrak{g} \to C^1$ be the operator given by the following formula:

(4.8)
$$\partial x(\delta) = [\delta, x], \quad \forall x \in \mathfrak{g}, \delta \in \mathfrak{m}.$$

Remark 4.2. As a matter of fact, one can look on $\mathfrak{g}$ and C^1 as on the spaces of 0-cochains and 1-cochains, respectively, on $\mathfrak{m}$ with values in $\mathfrak{g}$. Moreover, the operator ∂ is exactly the coboundary operator associated with the adjoint representation $\mathrm{ad} : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{g})$.

Recall also that the group $\mathcal{G}_+$ acts naturally on the space C^1 as follows:

(4.9)
$$(A.c)(\delta) = (\mathrm{Ad}A)c\Big(\big((\mathrm{Ad}A^{-1})\delta\big)_{-1}\Big), \quad a \in \mathcal{G}_+, \delta \in \mathfrak{m},$$

where $\big((\mathrm{Ad}A^{-1})\delta\big)_{-1}$ is a degree -1 component of $(\mathrm{Ad}A^{-1})\delta$ with respect to the grading of $\mathfrak{g}$. Obviously, this action restricts to the action on C^1_+.

Finally, let $\mathfrak{u}_+(\mathfrak{m}) = \bigoplus_{k \geq 0} \mathfrak{u}_k$ and let $U_+(\mathfrak{m})$ be the subgroup of the group of symmetries of the curve $t \mapsto \{e^{t\delta}V^i\}_{i=-1,\ldots,-\mu}$ preserving the filtration $\{V^i\}_{i=-\mu}^{-1}$. Note that $\mathfrak{u}_+(\mathfrak{m})$ is the Lie algebra of $U_+(\mathfrak{m})$.

A subspace $\mathcal{N}$ complementary to the space $\mathrm{Hom}(\mathfrak{m}, \mathfrak{u}_+(\mathfrak{m})) + (\mathrm{Im}\,\partial \cap C^1_+)$ in C^1_+ is called a *normalization condition*. We say that *an Ehresmann connection* $\mathcal{H}$ *on a subbundle* P *of* $\widehat{P}_0$ *satisfies the normalization condition* $\mathcal{N}$ if the positive part of the structure function of the pair $(\mathcal{P}, \mathcal{H})$ takes values in $\mathcal{N}$.

Our main theorem can be formulated as follows:

Theorem 4.1. *Fix a normalization condition* $\mathcal{N}$. *Then for a curve of flags* (3.1) *with constant symbol with representative* $\mathfrak{m}$ *there exists a unique quasi-principal subbundle* P *of the bundle* $\widehat{P}_0$ *with symbol* $\mathfrak{u}_+(\mathfrak{m})$ *and a unique Ehresmann connection* $\mathcal{H}$ *on it such that this connection satisfies the normalization condition* $\mathcal{N}$. *If, in addition, the space* $\mathcal{N}$ *is invariant with respect to the natural action of the subgroup* $U_+(\mathfrak{m})$ *on* C^1_+, *then the bundle* P *is the principal bundle with the structure group* $U_+(\mathfrak{m})$.

The pair $(P, \mathcal{H})$ from the previous theorem is called *the bundle of moving frames canonically associated with the curve* (3.1) *via the normalization condition* $\mathcal{N}$ and the Ehresmann connection $\mathcal{H}$ is called *the canonical Ehresmann connection on* P *associated with the curve* (3.1) *via the normalization condition* $\mathcal{N}$.

Theorem 4.1 is proved in the next section. As a direct consequence, we have that two curves $\{\Lambda_i\}_{i=-\mu}^{-1}$ and $\{\widetilde{\Lambda}_i\}_{i=-\mu}^{-1}$ of flags with constant symbol $\mathfrak{m}$ are equivalent through $A \in G$ if and only if for any moving frame $t \mapsto \psi(t)$ of the pair (the bundle P, the Ehresmann connection $\mathcal{H}$) canonically associated with the curve $\{\Lambda_i\}_{i=-\mu}^{-1}$ via $\mathcal{N}$ the curve $t \mapsto A \circ \psi(t)$ is the moving frame of the pair (the bundle $\widetilde{P}$, the Ehresmann connection $\widetilde{\mathcal{H}}$) canonically associated with the curve $\{\widetilde{\Lambda}_i\}_{i=-\mu}^{-1}$ via $\mathcal{N}$.

Remark 4.3. Note that in the case when the normalization condition $\mathcal{N}$ is invariant with respect to the natural action of the subgroup $U_+(\mathfrak{m})$ on C^1_+, the canonical Ehresmann connection is not a principal connection on the corresponding $U_+(\mathfrak{m})-$ principal bundle in general. It is a principal connection if the first algebraic prolongation $\mathfrak{u}_1$ is equal to 0.

4.5. **On geometry of parametrized curves of flags.** Similar results can be obtained for equivalence problem for *parametrized* curves of flags with respect to the action of group G. There are only three modifications. The first modification is that the symbol of a parametrized curve is

an orbit of an element (and not a line) in $\mathfrak{g}_{-1}$ with respect to the adjoint action of $\mathcal{G}_+$. Then we fix a representative δ of this orbit. The second modification is in the definition of the algebra $\mathfrak{u}_0$ and of the bundle $\widehat{P}_0$. Let $\delta \in \mathfrak{g}_{-1}$. Set $\mathfrak{u}_{-1} = \mathbb{K}\delta$ and

$$(4.10) \qquad\qquad \mathfrak{u}_0 := \{X \in \mathfrak{g}_0 : [X, \delta] = 0\}.$$

The spaces $\mathfrak{u}_k$ for $k > 0$ are defined recursively, using (4.1). The third modification is in the definition of the bundle $\widehat{P}_0$ (the zero degree reduction of the bundle $P|_{\Lambda(\cdot)}$). Here $\widehat{P}_0$ should be the subbundle of $\widehat{P}|_{\Lambda(\cdot)}$ with the fiber $\widehat{P}_0(t)$ over the point $\{\Lambda_i(t)\}_{i=-\mu}^{-1}$ consisting of all $A \in \widehat{P}_{\Lambda(t)}$ such that

$$\delta = \left(A^{-1} \circ \delta_t \circ A\right)_{-1}.$$

With these modifications all other constructions and Theorem 4.1 are valid.

4.6. On geometry of submanifolds of constant type in flag varieties.
Similar theory can be constructed for equivalence problem for submanifolds of dimension l of $\mathcal{O}$ with respect to the action of the group G. Again we consider submanifolds compatible with respect to differentiation. A submanifold is compatible with respect to differentiation if any smooth curve on it is compatible with respect to differentiation. As in the case of curves of flags the tangent space to an l-dimensional submanifold $S \subset \mathcal{O}$ at a point Λ can be identified with an orbit of an l-dimensional subspace $\mathfrak{m}$ (of $\mathfrak{g}$) belonging to $\mathfrak{g}_{-1}$ with respect to the natural action of $\mathcal{G}_+$ on l-dimensional subspaces of of $\mathfrak{g}_{-1}$, which is called the *symbol of a submanifold $S \subset \mathcal{O}$ at Λ*. This subspace can be taken as a linear span of all elements of type $\left(A^{-1} \circ \delta_t \circ A\right)_{-1}$, where A is a fixed element of the fiber $\widetilde{P}_\Lambda$ and δ_t are elements of $\mathrm{gr}_\Lambda W$ corresponding to all possible smooth curves in S, passing through Λ. Note that this subspace must be a commutative subalgebra of $\mathfrak{g}$. The latter condition follows from the involutivity of the tangent bundle to the submanifold S. Moreover, the word "commutative" can be removed, because any subalgebra of the graded Lie algebra $\mathfrak{g}$ that belongs to $\mathfrak{g}_{-1}$ must be commutative.

Further, one can define the bundle $\widehat{P}_0$ over the submanifold S as in subsection 4.2, the notion of a quasi-principal subbundle of the bundle $\widehat{P}_0$ as in subsection 4.3, an Ehresmann connection of P i.e. a rank l distribution $\mathcal{H}$ transversal to its fibers, and a moving frame of the pair $(P, \mathcal{H})$ as in subsection 4.4. The structure function of the $(P, \mathcal{H})$ can be defined by the relation (4.6). Note that since $\mathfrak{m}$ is abelian, the structure function at any point $\psi \in P$ takes its values in the space Z^1 of 1-cocycles (with respect to the coboundary operator associated with the adjoint representation $\mathrm{ad} : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{g})$, see Remark 4.2). Therefore, to get the generalization of Theorem 4.1 to submanifolds with a constant symbol in flag varieties (with literally the same proof as in section 5 below) one has to replace C_+^1 by Z_+^1 in the formulation of this theorem, where Z_+^1 is the subspace of cocycles in C_+^1. It is clear that in the case of curves $Z^1 = C^1$.

Finally note that one can consider more general situation of filtered submanifolds of constant type in flag variety, which is encoded by fixing a graded subalgebra in a negative part of $\mathfrak{g}$ (not necessary belonging to $\mathfrak{g}_{-1}$). This point of view is considered in the recent preprint [12].

5. Proof of Theorem 4.1

We construct a decreasing sequence of fiber subbundles of the bundle $\widehat{P}_0$ by induction.

First let us introduce some notations. If $\mathcal{N}$ is the space from the formulation of Theorem 4.1, denote

$$(5.1) \qquad\qquad \mathcal{N}_k = \mathcal{N} \cap C_k^1, \quad k \in \mathbb{N}$$

Then obviously $\mathcal{N} = \bigoplus_{k \in \mathbb{N}} \mathcal{N}_k$.

Note also that there is a natural mappings $\mathrm{pr}_k : C_k^1 \to \mathrm{Hom}(\mathfrak{m}, \mathfrak{g}_{k-1}/\mathfrak{u}_{k-1})$ such that for any $c_k \in C_k^1$ and $\delta \in \mathfrak{m}$ we set $\mathrm{pr}_k(c_k)(\delta)$ to be the equivalence class of $c_k(\delta)$ in $\mathfrak{g}_{k-1}/\mathfrak{u}_{k-1}$. Let

$$(5.2) \qquad \overline{\mathcal{N}}_k = \mathrm{pr}_k(\mathcal{N}_k).$$

Further, let $\mathfrak{g}_k$ be as in (3.4). Then the image of the restriction of the operator ∂ to the subspace $\mathfrak{g}_k$ belongs to C_k^1. We denote this restriction by ∂_k, $\partial_k : \mathfrak{g}_k \to C_k^1$. Finally, let

$$(5.3) \qquad \bar{\partial}_k = \mathrm{pr}_k \circ \partial_k.$$

Note that by constructions the space $\mathfrak{u}_k$ defined by (4.1) is nothing but the kernel of the map $\bar{\partial}_k$, $\mathfrak{u}_k = \ker \bar{\partial}_k$.

5.1. **First degree normalization.** First take some Ehresmann connection on the bundle $\widehat{P}_0$ and let c_1 be the degree 1 component of the structure function of the bundle $\widehat{P}_0$ with this Ehresmann connection. If one takes another Ehresmann connection on $\widehat{P}_0$ and the corresponding degree 1 component $\tilde{c}_1$ of the structure function of $\widehat{P}_0$ with this new Ehresmann connection, then

$$(5.4) \qquad \tilde{c}_1(\psi)(\delta) - c_1(\psi)(\delta) \in \mathfrak{u}_0 \quad \forall \psi \in \widehat{P}_0 \text{ and } \delta \in \mathfrak{m}$$

In other words, the map $\bar{c}_1 : \widehat{P}_0 \to \mathrm{Hom}(\mathfrak{m}, \mathfrak{g}_0/\mathfrak{u}_0)$, defined by $\bar{c}_1(\psi) = \mathrm{pr}_1 \circ c_1(\psi)$, is independent of a choice of an Ehresmann connection on $\widehat{P}_0$.

Now take ψ_1 and ψ_2 from the same fiber $\widehat{P}_0(t)$ such that $\Pi_0^t(\psi_1) = \Pi_0^t(\psi_2)$ and study how $\bar{c}_1(\psi_1)$ and $\bar{c}_1(\psi_2)$ are related. As before given $A \in \mathfrak{gl}(V)$ denote by $(A)_k$ the component of degree k of A (w.r.t. the splitting $\mathfrak{gl}(V) = \bigoplus_{k \in \mathbb{Z}} \mathfrak{gl}(V)_k$). Set

$$(5.5) \qquad f_{\psi_1, \psi_2}^1 := \left(\psi_1^{-1} \circ \psi_2 \right)_1.$$

It is not hard to see that $f_{\hat{\varphi}, \hat{\varphi}_1}^1 \in \mathfrak{g}_1$, where $\mathfrak{g}_1$ is defined by (3.4). In opposite direction, for fixed $\psi_1 \in \widehat{P}_0(t)$ and any $f \in \mathfrak{g}_1$ there exists $\psi_2 \in \widehat{P}_0(t)$ such that $\Pi_0^t(\psi_1) = \Pi_0^t(\psi_2)$ and $f = f_{\psi_1, \psi_2}^1$.

Lemma 5.1. *The following identity holds*

$$(5.6) \qquad \bar{c}_1(\psi_2) = \bar{c}_1(\psi_1) + \bar{\partial}_1(f_{\psi_1, \psi_2}^1).$$

Proof. First note that

$$(5.7) \qquad f_{\psi_1, \psi_2}^1 = \left(\Pi_0^t(\psi_1) \right)^{-1} \circ \left(\Pi_1^t(\psi_2) - \Pi_1^t(\psi_1) \right).$$

or, equivalently,

$$(5.8) \qquad \psi_2(v) = \psi_1(v) + \Pi_0^t(\psi_1) \circ f_{\psi_1, \psi_2}^1(v) \bmod \Lambda_{i+2}(t) \quad \forall v \in V_i.$$

Let $\psi_i(\tau)$, $i = 1, 2$ be a section of $\widehat{P}_0$ such that $\psi_i(t) = \psi_i$ and

$$(5.9) \qquad \left(\psi_i^{-1} \circ \psi_i'(t) \right)_{-1} = \delta$$

Then one can write the identity analogous to (5.8) for any τ. Differentiating it at $\tau = t$ and using (5.9) again we get

$$(5.10) \qquad \psi_2'(t)(v) = \psi_1'(t)(v) + \Pi_0^t(\psi_1) \circ \delta \circ f_{\psi_1, \psi_2}^1(v) \bmod \Lambda_{i+1}(t) \quad \forall v \in V_i.$$

Applying ψ_2^{-1} to both sides and using (5.9) again one gets

(5.11) $\qquad \psi_2^{-1} \circ \psi_2'(t)(v) = \psi_2^{-1} \circ \psi_1'(t)(v) + \delta \circ f_{\psi_1,\psi_2}^1(v) \bmod \Lambda_{i+1}(t) \quad \forall v \in V_i.$

Representing $\psi_2^{-1} \circ \psi_1'(t)$ as $\psi_2^{-1} \circ \psi_1 \circ \psi_1^{-1} \circ \psi_1'(t)$ and taking into account that

$$\psi_2^{-1} \circ \psi_1 = Id - f_{\psi_1,\psi_2}^1 + \text{operators of degree} \geq 2,$$
$$\psi_1^{-1} \circ \psi_1'(t) \equiv \delta + \bar{c}_1(\psi_1) + \text{operators of degree} \geq 1 \mod \mathfrak{u}_0.$$

we get that

$$\left(\psi_2^{-1} \circ \psi_1'(t) \right)_0 = \bar{c}_1(\psi_1) - f_{\psi_1,\psi_2}^1 \circ \delta \mod \mathfrak{u}_0.$$

Finally, comparing 0-degree components of identity (5.11) we get the required identity (5.6). $\square$

Now take the space $\mathcal{N}_1$ as in (5.1). By our assumptions on $\mathcal{N}$ we have

(5.12) $\qquad\qquad\qquad\qquad \text{Hom}(\mathfrak{m}, \mathfrak{g}_0/\mathfrak{u}_0) = \text{Im}\, \bar{\partial}_1 \oplus \overline{\mathcal{N}}_1.$

Let $\widehat{P}_1$ be the subbundle of the bundle $\widehat{P}_0$ consisting of all $\psi \in \widehat{P}_0$ such that $\bar{c}_1(\psi) \in \overline{\mathcal{N}}_1$. From formula (5.6) and splitting (5.12) it follows that the bundle $\widehat{P}_1$ is not empty. Moreover, ψ_1 and ψ_2 from the same fiber $\widehat{P}_1(t)$ satisfy $\Pi_0^t(\psi_1) = \Pi_0^t(\psi_2)$ if and only if $f_{\psi_1,\psi_2}^1 \in \ker \bar{\partial}_1 = \mathfrak{u}_1$.

We also denote by P_1 the bundle over our curve with the fiber $P_1(t)$ over the point $\{\Lambda_i(t)\}_{i=-\mu}^{-1}$ equal to the image of the corresponding fiber $\widehat{P}_1(t)$ of the bundle $\widehat{P}_1$ under the map Π_1^t. By above P_1 can be considered as an affine bundle over P_0 such that its fibers are affine spaces over the vector space $\mathfrak{u}_1$. The projection of the bundle is induced by the maps Π_0^t.

Note that in the constructions of the bundles $\widehat{P}_1$ and P_1 we did not use the whole information on the space $\mathcal{N}_1$, but on its image under the (non-injective) map pr_1 only. So, from the knowledge of the whole space $\mathcal{N}_1$ we should get additional restrictions on the bundle of moving frames we are looking for. Indeed, by our assumptions on $\mathcal{N}$ we have the following splitting

(5.13) $\qquad\qquad\qquad\qquad (\text{pr}_1)^{-1}(\overline{\mathcal{N}}_1) = \mathcal{N}_1 \oplus \text{Hom}(\mathfrak{m}, \mathfrak{u}_0).$

From this splitting and relation (5.4) it follows that one can choose an Ehresmann connection on $\widehat{P}_1$ such that the degree 1 component c_1 of the structure equation of $\widehat{P}_1$ with this Ehresmann connection belongs to $\mathcal{N}_1$. Moreover, any such Ehresmann connection on $\widehat{P}_1$ induces the unique Ehresmann connection on the bundle P_0.

5.2. Higher degree normalizations: the induction step.
Given $i > 0$ assume that there exists a unique series of bundles $\widehat{P}_k$, $1 \leq k \leq i$ over our curve such that $\widehat{P}_0$ is as above and for each $1 \leq k \leq i$ the bundle $\widehat{P}_k$ satisfies the following properties:

(A1) $\widehat{P}_k$ is a subbundle of $\widehat{P}_{k-1}$;

(A2) if c is the structure function of $\widehat{P}^k$ (endowed with some Ehresmann connection) and c_l is its degree l component, then

(5.14) $\qquad\qquad\qquad\qquad \text{pr}_l \circ c_l \in \overline{\mathcal{N}}_l \quad \forall 1 \leq l \leq k$

(A3) if P_k, $0 \leq k \leq i$, is the bundle over our curve with the fiber $P_k(t)$ over the point $\{\Lambda_j(t)\}_{j=-\mu}^{-1}$ equal to the image of the corresponding fiber $\widehat{P}_k(t)$ of the bundle $\widehat{P}_k$ under the map Π_k^t, then P_k can be considered as an affine bundle over P_{k-1} such that its fibers are affine spaces over the vector space $\mathfrak{u}_k$. The projection of this bundle is induced by the maps Π_{k-1}^t;

(A4) there exists an Ehresmann connection on the bundle $\widehat{P}_k$ such that the structure function c satisfies

$$(5.15) \qquad c_l \in \mathcal{N}_l \quad \forall\, 1 \le l \le k.$$

Moreover, any such Ehresmann connection on $\widehat{P}_k$ induces the unique Ehresmann connection on the bundle P_{k-1}.

Now let us construct the subbundle $\widehat{P}_{i+1}$ of $\widehat{P}_i$, corresponding to the normalization of degree $i+1$. For this first take some Ehresmann connection on $\widehat{P}_i$ satisfying (5.15) (with $k = i$) and let c_{i+1} be the degree $i+1$ component of the structure function of the bundle $\widehat{P}_i$ with this Ehresmann connection. If one takes another Ehresmann connection on $\widehat{P}_i$ satisfying (5.15) (with $k = i$) and the corresponding degree $i+1$ component $\tilde{c}_{i+1}$ of the structure function of $\widehat{P}_i$ with this new Ehresmann connection, then

$$(5.16) \qquad \tilde{c}_{i+1}(\psi)(\delta) - c_{i+1}(\psi)(\delta) \in \mathfrak{u}_i \quad \forall\, \psi \in \widehat{P}_i \text{ and } \delta \in \mathfrak{m}.$$

In other words, the map $\bar{c}_{i+1} : \widehat{P}_i \to \operatorname{Hom}(\mathfrak{m}, \mathfrak{g}_i/\mathfrak{u}_i)$, defined by $\bar{c}_{i+1}(\psi) = \operatorname{pr}_{i+1} \circ c_{i+1}(\psi)$, is independent of a choice of an Ehresmann connection on $\widetilde{P}_i$.

Now take ψ_1 and ψ_2 from the same fiber $\widehat{P}_i(t)$ of the bundle P_i such that $\Pi_i^t(\psi_1) = \Pi_i^t(\psi_2)$ and study how $\bar{c}_{i+1}(\psi_1)$ and $\bar{c}_{i+1}(\psi_2)$ are related. Set

$$(5.17) \qquad f_{\psi_1,\psi_2}^{i+1} := \left(\psi_1^{-1} \circ \psi_2 \right)_{i+1}.$$

It is not hard to see that $f_{\psi_1,\psi_2}^{i+1} \in \mathfrak{g}_{i+1}$, where $\mathfrak{g}_{i+1}$ is defined by (3.4). In opposite direction, for fixed $\psi_1 \in \widehat{P}_i(t)$ and any $f \in \mathfrak{g}_{i+1}$ there exists $\psi_2 \in \widehat{P}_i(t)$ such that $\Pi_i^t(\psi_1) = \Pi_i^t(\psi_2)$ and $f = f_{\psi_1,\psi_2}^{i+1}$. By the complete analogy with the proof of Lemma 5.1 one gets the following identity

$$(5.18) \qquad \bar{c}_{i+1}(\psi_2) = \bar{c}_{i+1}(\psi_1) + \bar{\partial}_{i+1}(f_{\psi_1,\psi_2}^{i+1}).$$

Further take the space $\mathcal{N}_{i+1}$ as in (5.1). By our assumptions on $\mathcal{N}$ we have

$$(5.19) \qquad \operatorname{Hom}(\mathfrak{m}, \mathfrak{g}_i/\mathfrak{u}_i) = \operatorname{Im} \bar{\partial}_{i+1} \oplus \overline{\mathcal{N}}_{i+1}.$$

Let $\widehat{P}_{i+1}$ be the subbundle of the bundle $\widehat{P}_i$ consisting of all $\psi \in \widehat{P}_i$ such that $\bar{c}_{i+1}(\psi) \in \overline{\mathcal{N}}_{i+1}$. From formula (5.18) and splitting (5.19) it follows that the $\widehat{P}_{i+1}$ is not empty. Moreover, ψ_1 and ψ_2 from to the same fiber $\widehat{P}_{i+1}(t)$ satisfy $\Pi_i^t(\psi_1) = \Pi_i^t(\psi_2)$ if and only if $f_{\psi_1,\psi_2}^{i+1} \in \ker \bar{\partial}_i = \mathfrak{u}_{i+1}$.

We also denote by P_{i+1} the bundle over our curve with the fiber $P_{i+1}(t)$ over the point $\{\Lambda_j(t)\}_{j=-\mu}^{-1}$ equal to the image of the corresponding fiber $\widehat{P}_{i+1}(t)$ of the bundle $\widehat{P}_{i+1}$ under the map Π_{i+1}^t. By above P_{i+1} can be considered as an affine bundle over P_i such that its fibers are affine spaces over the vector space $\mathfrak{u}_{i+1}$. The projection of the bundle is induced by the maps Π_i^t. So, the bundles $\widehat{P}_{i+1}$ and P_{i+1} satisfy conditions (A1)-(A3) above (for $k = i+1$).

To construct an Ehresmann connection $\widehat{P}_{i+1}$, satisfying condition (A4), consider the splitting

$$(5.20) \qquad (\operatorname{pr}_{i+1})^{-1}(\overline{\mathcal{N}}_{i+1}) = \mathcal{N}_{i+1} \oplus \operatorname{Hom}(\mathfrak{m}, \mathfrak{u}_i).$$

From this splitting and relation (5.16) it follows that one can choose an Ehresmann connection on $\widehat{P}_{i+1}$ such that the degree $i+1$ component c_{i+1} of the structure equation of $\widehat{P}_{i+1}$ with this Ehresmann connection belongs to $\mathcal{N}_{i+1}$. Moreover, any such Ehresmann connection on $\widehat{P}_{i+1}$ induces the unique Ehresmann connection on the bundle P_i. This completes the induction step.

Since $\mathfrak{u}_k = 0$ for $k \geq \mu$ and the maps Π_i^t are identities for $i \geq \mu - 1$, the sequence of bundles $\{\widehat{P}_i\}_{i \geq 0}$ stabilizes from $i = \mu - 1$, i.e. $\widehat{P}_i = \widehat{P}_{i+1}$ for $i \geq \mu - 1$, and also $P_i = \widehat{P}_i$ for $i \geq \mu - 1$. Moreover, the bundle P_μ $(= \widehat{P}_\mu)$ is endowed with the unique Ehresmann connection such that its structure function satisfies (5.15) with $k = \mu$.

We claim that the bundle P_μ with the constructed Ehresmann connection is the bundle we are looking for in Theorem 4.1. By construction the positive part of the structure function of P_μ takes values in $\mathcal{N}$. Let us check that the bundle P_μ has symbol $\mathfrak{u}_+(\mathfrak{m})$. By the previous constructions we have the sequence of the bundles $\{P_i\}_{i=0}^\mu$ such that P_{i+1} is a bundle over P_i. Hence, P_μ can be considered as a bundle over each P_i with $0 \leq i < \mu$. Take $\psi \in P_\mu$ and consider the fibers of P_μ considered as a bundle of P_i for each $0 \leq i < \mu$. The tangent spaces to this fibers at ψ define the filtration on $T_\psi P_\mu(t)$. Let ω and L_ψ be as in (4.3) and (4.4) respectively. The images of the spaces of the filtration on $T_\psi P_\mu(t)$ under the map $\omega(\psi)$ define the filtration on the space L_ψ . By our constructions this filtration coincides with the filtration induced on L_ψ from $\mathfrak{gl}(V)$. Moreover, by property (A3) the graded space $\mathrm{gr}\, L_\psi$ of L_ψ coincides with $\mathfrak{u}_+(\mathfrak{m})$. In other words, the bundle P_μ has constant symbol $\mathfrak{u}_+(\mathfrak{m})$. So, P_μ satisfies all conditions of our theorem. Moreover, all conditions we imposed on the structure function of P_μ during the proof were necessary, which proves the uniqueness of the constructed bundle and completes the proof of the first part of our theorem.

Note that by our constructions the bundle P_μ (without a specified Ehresmann connection on it) can be uniquely determined by the subspace $\overline{\mathcal{N}} = \bigoplus_{k>0} \overline{\mathcal{N}}_k$ of $\mathrm{Hom}\big(\mathfrak{m}, \mathfrak{g}_+/\mathfrak{u}_+(\mathfrak{m})\big)$: the bundle P_μ consists of all $\psi \in \widehat{P}_0$ such that $\bar{c}_i(\psi) \in \overline{\mathcal{N}}_i$ for all $1 \leq i \leq \mu$. The natural action of $U_+(\mathfrak{m})$ on C_+^1 defined by (4.9) induces the action on $\mathrm{Hom}\big(\mathfrak{m}, \mathfrak{g}_+/\mathfrak{u}_+(\mathfrak{m})\big)$. Now assume that $\mathcal{N}$ is invariant with respect to the action of $U_+(\mathfrak{m})$ on C_+^1. Then the corresponding space $\overline{\mathcal{N}}$ is invariant with respect to the action of $U_+(\mathfrak{m})$ on $\mathrm{Hom}\big(\mathfrak{m}, \mathfrak{g}_+/\mathfrak{u}_+(\mathfrak{m})\big)$. This implies that if $\psi \in P_\mu$ then $\psi \circ B \in P_\mu$ for any $B \in U_+(\mathfrak{m})$, which implies that the bundle P_μ is a principal $U_+(\mathfrak{m})$-bundle. This completes the proof of the second part of our theorem.

6. Refinement procedure

In this section we discussed the so-called refinement procedure in the cases of the standard representations of the classical groups and the case $G = \mathrm{Aff}(W)$ which also sheds light on the condition of compatibility with respect to differentiation.

6.1. **The case $G = SL(W)$, $GL(W)$, and $\mathrm{Aff}(W)$.** The constructions of this subsection work for any group G such that any orbit $\mathcal{O}$ of flags is compatible with respect to some grading of the corresponding Lie algebra g. Starting with a curve of flags (3.1) compatible with respect to differentiation, we can build a finer curve of flags compatible with respect to differentiation. For this first given a curve $t \mapsto L(t)$ of subspaces of W (parametrized somehow), i.e. a curve in a certain Grassmannian of W, denote by $C(L)$ the canonical bundle over L: the fiber of $C(L)$ over the point $L(t)$ is the vector space $L(t)$. Let $\Gamma(L)$ be the space of all sections of $C(L)$. Set $L^{(0)}(t) := L(t)$ and define inductively

$$L^{(j)}(t) = \mathrm{span}\{\frac{d^k}{d\tau^k}\ell(t) : \ell \in \Gamma(L), 0 \leq k \leq j\}$$

for $j \geq 0$. The space $L^{(j)}(t)$ is called the *jth extension* or the *jth osculating subspace* of the curve L at the point t. The compatibility of the curve of flags (3.1) with respect to differentiation is equivalent to the condition $\Lambda_i^{(1)}(t) \subset \Lambda_{i-1}(t)$ for any $-\mu + 1 \leq i \leq -1$ and for any t.

Further given a subspace L in W denote by $L^\perp$ the annihilator of L in the dual space W^*: $L^\perp = \{p \in W^* : \langle p, v \rangle = 0, \forall v \in L\}$. Set $L^{(-j)}(t) = \left((L(t)^\perp)^{(j)}\right)^\perp$ for $j \geq 0$. For any curve $t \mapsto L(t)$ in a Grassmannian we get the nondecreasing filtration $\{L^{(j)}(t)\}_{j \in \mathbb{Z}}$ of W. If dimensions $L^{(j)}(t)$ are independent of t for any $j \geq 0$, then the subspaces $L^{(j)}(t)$, $j < 0$, can be also defined inductively by $L^{(j)}(t) = \mathrm{Ker} \frac{d}{dt} L^{(j+1)}(t)$.

Now take a germ of the curve (3.1) at a point t and assume that it is compatible with respect to differentiation. Assume that there exists a neighborhood $\mathcal{U}$ of t such that one of the following assumptions hold:

(B1) For $\tau \in \mathcal{U}$ the dimension of subspaces $\Lambda_i^{(1)}(\tau)$ is constant, the subspace $\Lambda_i^{(1)}(\tau)$ is strictly between $\Lambda_i(t)$ and $\Lambda_{i-1}(t)$, and one has the inclusion $\Lambda_i^{(2)}(\tau) \subset \Lambda_{i-1}(\tau)$;

(B2) For $\tau \in \mathcal{U}$ the dimension of the space $\Lambda_i^{(-1)}(\tau)$ is constant, the subspace $\Lambda_i^{(-1)}(\tau)$ is strictly between $\Lambda_{i+1}(t)$ and $\Lambda_i(t)$, and one has the inclusion $\Lambda_{i+1}(\tau) \subset \Lambda_i^{(-2)}(\tau)$ (or, equivalently, $\Lambda_{i+1}^{(1)}(\tau) \subset \Lambda_i^{(-1)}(\tau)$).

Note that for $i = \mu + 1$ the condition $\Lambda_i^{(2)}(\tau) \subset \Lambda_{i-1}(\tau)$ holds automatically in (B1) and for $i = -1$ the condition $\Lambda_{i+1}(\tau) \subset \Lambda_i^{(-2)}(\tau)$ holds automatically in (B2).

If assumption (B1) holds, then we can obtain a new germ of a curve of flags by inserting in (3.1) the space $\Lambda_i^{(1)}(\tau)$ between $\Lambda_i(\tau)$ and $\Lambda_{i-1}(\tau)$ for any $\tau \in \mathcal{U}$. We call such operation *elementary refinement of the first kind*. In the same way if assumption (B2) holds, then we can obtain a new germ of a curve of flag by inserting in (3.1) the space $\Lambda_i^{(-1)}(\tau)$ between $\Lambda_{i+1}(\tau)$ and $\Lambda_i(\tau)$ for any $\tau \in \mathcal{U}$. We call such operation *elementary refinement of the second kind*. In both cases we renumber the indices of subspaces in the obtained flags from -1 to $-\mu - 1$. The germ of a curve of flags t is called a *refinement of the germ of the curve* (3.1) *at t*, if it can be obtained from this germ by a sequence of elementary refinements (either of the first or the second kind).

By construction any refinement preserves the property of compatibility with respect to differentiation. Besides, if two curves are $GL(W)$-equivalent, then the corresponding refinements are $GL(W)$-equivalent. Further, the refinement procedure defines a partial order on the set of all germs of smooth curves of flags in W, which also induces a partial order on the set of all refinements of the germ of the curve (3.1) at t. In general a curve of flags admits several different maximal refinements with respect to this partial order. Finally, for a curve $t \mapsto L(t)$ in a Grassmannian of W consider the corresponding trivial curve of flags $t \mapsto \{0 \subset L(t) \subset W\}$. Then it is easy to show that in a generic point the germ of such curve of flags has the unique maximal refinement, which is exactly the curve $t \mapsto \{L^{(j)}(t)\}_{j \in \mathbb{Z}}$ of all osculating subspaces of the curve L (truncated and renumbered in an obvious way). The corresponding map $\delta_t \in \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}\left(L^j(t)/L^{j-1}(t), L^{j+1}(t)/L^j(t)\right)$

satisfies the following properties: the restriction of δ_t to $L^j(t)/L^{j-1}(t)$ is surjective for $j < 0$ and δ_t sends $L^j(t)/L^{j-1}(t)$ *onto* $L^{j+1}(t)/L^j(t)$ for $j \geq 0$. Hence the symbol of the curve $t \mapsto \{L^{(j)}(t)\}_{j \in \mathbb{Z}}$ satisfies similar properties. Note that the symbol of a curve of flags at a given point is in essence the tangent line to a curve. The symbol of the refined curve of flags contains an important information about the jet space of higher order of the original curve. The procedure of refinement allows us to identify a curve as a member of a more specific class of curves, so applying our procedure of construction of the canonical bundle of moving frames to a refinement will usually lead to a finer invariants. In particular, the dimension of the algebraic prolongation of the symbol of the refined curve (which in turn is equal to the dimension of the canonical bundle of moving frames for

the refined curve) might be significantly smaller than the dimension of the corresponding objects for the original curve.

Example 6.1. *(Non-degenerate curve in projective space)* Consider a curve $t \mapsto L(t)$ in a projective space $\mathbb{P}W$ of an $n+1$-dimensional vector space W, i.e. a curve of lines in W. Without making a refinement procedure the symbol at a generic point of a curve, which does not degenerate to a point, is a line of degree -1 endomorphisms of the corresponding graded space having rank 1 and the flat curve corresponds to a curve of lines in W depending linearly on a parameter (or, equivalently, it is a projective line linearly embedded to $\mathbb{P}W$). Obviously, the universal algebraic prolongation coincides with the subalgebra of $\mathfrak{gl}(W)$ preserving the plane, generated by the lines of a flat curve. Now consider so-called non-degenerate curves in $\mathbb{P}W$, i.e. curves which do not lie in any proper subspace of $\mathbb{P}W$. For such curve the nth osculating space at a generic point t is equal to the whole W, $L^{(n)}(t) = W$. The maximal refinement of the curve $t \mapsto \{0 \subset L(t) \subset W\}$ is the curve of complete flags

$$t \mapsto \{0 \subset L(t) \subset L^{(1)}(t) \subset \ldots L^{(n)}(t) = W\},$$

as in Example 3.1 above. As already mentioned there, the symbol of such curve is a line of degree -1 endomorphisms of the corresponding graded spaces, generated by an endomorphism which has the matrix equal to a Jordan nilpotent block in some basis and the curve in projective space corresponding to a flat curve of complete flags is the curve of osculating subspaces of a rational normal curve in the projective space $\mathbb{P}W$. Finally, from Theorem 8.1 below (see also the sentence after it) it follows that the universal algebraic prolongation of this symbol is isomorphic to $\mathfrak{gl}_2$ (or $\mathfrak{sl}_2$ in the case $G = SL(W)$), which also follows from the well-known fact that the algebra of infinitesimal symmetries of a rational normal curve is isomorphic to $\mathfrak{gl}_2$ (or $\mathfrak{sl}_2$ in the case $G = SL(W)$). In this case one can show that the normalization condition can be chosen to be invariant with respect to the natural action of the subgroup $U_+(\mathfrak{m})$ (which is isomorphic in this case to the group the upper-triangular matrices) on C_+^1 and in this way the classical complete system of invariants of curves in projective spaces, the Wilczynski invariants, can be constructed (see [16] or [10] for the detail). $\square$

6.2. **The case $G = Sp(W)$ or $G = CSp(W)$.** Here W is equipped with a symplectic form σ. In this case, by Proposition 2.2 our theory works for curves of *symplectic flags* as in Definition 2.1. Given a curve $t \mapsto L(t)$ in a Grassmannian of W denote by $t \mapsto L^{\angle}(t)$ the curve of subspaces of W such that the space $L^{\angle}(t)$ is the skew-symmetric complement of the space $L(t)$ with respect to the symplectic form σ. Starting with a germ of a curve (3.1) of symplectic flags compatible with respect to differentiation and applying an elementary refinement as in the previous subsection, we obtain non-symplectic flags in general. Therefore we need to modify the definition of an elementary refinement appropriately. Such modification is based on the following simple fact: if $t \mapsto L(t)$ is a curve of coisotropic or isotropic subspaces of W then $(L^{(1)})^{\angle}(t) = (L^{\angle})^{(-1)}(t)$. We say that a refinement of a curve of symplectic flag $t \mapsto \{\Lambda_i(t)\}_{i \in \mathbb{Z}}$ is an elementary symplectic refinement in one of the following 2 cases:

(C1) If condition (B1) holds with subspaces $\Lambda_i(\tau)$ being coisotropic, then an *elementary symplectic refinement of the first kind* consists of inserting $\Lambda_i^{(1)}(\tau)$ between $\Lambda_i(\tau)$ and $\Lambda_{i-1}(\tau)$ and inserting $(\Lambda_i^{\angle})^{(-1)}(\tau)$ between $(\Lambda_{i-1})^{\angle}(\tau)$ and $(\Lambda_i)^{\angle}(\tau)$ for any $\tau \in \mathcal{U}$;

(C2) If condition (B2) holds with subspaces $\Lambda_{i+1}(\tau)$ being coisotropic, then an *elementary symplectic refinement of the second kind* consists of inserting $\Lambda_i^{(-1)}(\tau)$ between $\Lambda_{i+1}(\tau)$ and $\Lambda_i(\tau)$ and inserting $(\Lambda_i^{\angle})^{(1)}(\tau)$ between $(\Lambda_i)^{\angle}(\tau)$ and $(\Lambda_{i+1})^{\angle}(\tau)$ for any $\tau \in \mathcal{U}$;

(C3) If $\Lambda_i(\tau)$ is isotropic and $\Lambda_{i-1}(\tau)$ is coisotropic (which implies by assumptions that $\Lambda_{i-1}(\tau) = \Lambda_i^{\angle}(\tau)$), then assume that $(\Lambda_i)^{(1)}(\tau)$ is isotropic, of constant dimension in $\mathcal{U}$, $\Lambda_i(\tau)$ is strictly contained in $(\Lambda_i)^{(1)}(\tau)$, and $\Lambda_i^{(2)} \subset \Lambda_{i-1}^{(-1)}(\tau)$. An *elementary symplectic refinement of the third kind* consists of inserting $\Lambda_i^{(1)}(\tau)$ and $\Lambda_{i-1}^{(-1)}(\tau)$ between $\Lambda_i(\tau)$ and $\Lambda_{i-1}(\tau)$ (if $\Lambda_i^{(1)}(\tau)$ and $\Lambda_{i-1}^{(-1)}(\tau)$ coincide they count as a one space).

By analogy with the previous subsection one can define a symplectic refinement as a composition of elementary symplectic refinement. By construction the resulting flag under a symplectic refinement is symplectic and compatible with respect to diferentiation. Also the refinement procedure defines a partial order on the set of all germs of smooth curves of symplectic flags in W. In general, similar to the previous subsection, a curve of symplectic flags admits several different maximal symplectic refinements with respect to this partial order. There is a unique maximal refinement for the germs at generic points of the following two types of curves of symplectic flags: $t \mapsto \{0 \subset L(t) \subset W\}$, where spaces $L(t)$ are Lagrangian, i.e. isotropic of dimension $\frac{1}{2} \dim W$, and $t \mapsto \{0 \subset L(t) \subset L^{\angle}(t) \subset W\}$, where spaces $L(t)$ are proper isotropic. In the first case we can use elementary refinement of the first kind only and the maximal refinement coincides with the flag associated with a curve in a Lagrangian Grassmannian, introduced in [41],[42]. In the second case one can use elementary refinements of the first and third kinds only.

6.3. The case $G = O(W)$ or $G = CO(W)$.
Here W is equipped with a non-degenerate symmetric form Q. The elementary refinement are defined in completely the same way as in the symplectic case: isotropic and coisotropic subspaces are taken with respect to the form Q and instead of skew-symmetric complement one takes orthogonal complements with respect to Q.

7. CLASSIFICATION OF SYMBOLS OF CURVES OF FLAGS WITH RESPECT TO CLASSICAL GROUPS

We describe all symbols of curves of flags with respect to $GL(W)$ (equivalently, $SL(W)$) , $Sp(W)$ (equivalently, $CSp(W)$), and $O(W)$ (equivalently, $CO(W)$. According to Remark 3.2 it can be used for any representation of classical groups. Essentially the developments of this section boil down to a little bit finer classification than the classification of nilpotent endomorphisms via the Jordan Normal Form, taking into account the grading of the vector space on which the endomorphism acts and the presence of a symplectic or symmetric form (see Remarks 7.1, 7.3, and 7.5 below)

7.1. The case of $GL(W)(SL(W))$.
Let δ_1 and δ_2 be degree -1 endomorphisms of the graded spaces V_1 and V_2, respectively. The direct sum $V_1 \oplus V_2$ is equipped with the natural grading such that its ith component is the direct sum of ith components of V_1 and V_2. The direct sum $\delta_1 \oplus \delta_2$ is the degree -1 endomorphism of $V_1 \oplus V_2$ such that the restriction of it to V_i is equal to δ_i for each $i = 1, 2$. A degree -1 endomorphism δ of a graded space V is called *indecomposable* if it cannot be represented as a direct sum of two degree -1 endomorphisms acting on nonzero graded spaces. Further, given two integers $r \leq s < 0$ let $V_{rs} = \bigoplus_{i=r}^{s} E_i$, where $\dim E_i = 1$ for every i, $r \leq i \leq s$, and let δ_{rs} be a degree -1 endomorphism of V_{rs} which sends E_i onto E_{i-1} for every i, $r < i \leq s$, and sends E_r to 0. Then clearly δ_{rs} is indecomposable.

The following theorem gives the classification of all symbols of curves of flags (both of unparametrized and parametrized) with respect to the General Linear group:

Theorem 7.1. *A degree -1 endomorphism δ of a graded space (indexed by negative integers) is conjugate to the direct sum of endomorphisms of type δ_{rs}. Moreover, for any two integers $r \leq s < 0$ the number of apperance of δ_{rs} in this direct sum is an invariant of the conjugacy class.*

Proof. We will proceed as in the classical proof of the Jordan Normal Form Theorem (see, for example, [22, chapter 3]). The only difference is that one has to take into account the grading on the ambient space V as well. Let δ be a degree -1 endomorphism of V. Then δ defines the additional filtration on V via generalized eigenspaces of different orders (corresponding to the unique eigenvalue 0), namely $\ker(\delta) \subset ker(\delta^2) \subset \ldots \subset \ker(\delta^{l+1}) = V$. In the first step choose a tuple of vectors in V consisting of homogeneous vectors (with respect to the grading on V) such that their images under the canonical projection to the factor space $\ker(\delta^{l+1})/\ker(\delta^l)$ constitute a basis of $\ker(\delta^{l+1})/\ker(\delta^l)$. Then take the images of vectors of this tuple under δ. Since δ is of degree -1, these images are homogeneous vectors in V belonging to $\ker(\delta^l)$. Complete the tuple of these images (if necessary) to a tuple of homogeneous vectors in $\ker(\delta^l)$ such that the images of vectors in this tuple under the canonical projection to the factor space $\ker(\delta^l)/\ker(\delta^{l-1})$ constitute a basis of $\ker(\delta^l)/\ker(\delta^{l-1})$. Continuing this process we will get a basis of V consisting of homogeneous vectors such that the matrix of the operator δ in this bases has a Jordan normal form. Each Jordan block corresponds to an indecomposable endomorphism of type δ_{rs}. This proves that δ is conjugate to the direct sum of endomorphisms δ_{rs}. The invariance of the number of appearance of δ_{rs} in this direct sum can be easily obtained from the above construction as well. $\qquad\square$

Remark 7.1. Note that in the basis $\{E_i\}_{i=r}^s$ the operator δ_{rs} corresponds to one Jordan block. Besides, it is important to keep the information not only about the size of this block but also about the highest (or/and the lowest) degree of the elements of this basis. Let us call such block the *weighted Jordan block with the weight* (r, s). Informally speaking, Theorem 7.1 says that any degree -1 endomorphism δ of a graded space is classified by the Jordan normal form with weighted Jordan blocks, where the weights count. $\square$

The previous theorem also shows that *the endomorphisms δ_{rs} are the only indecomposable degree -1 endomorphisms, up to a conjugation, of a graded space (indexed by negative integers).*

7.2. **Symplectic case.** Let $\frac{1}{2}\mathbb{Z} := \{\frac{i}{2} : i \in \mathbb{Z}\}$ and $\frac{1}{2}\mathbb{Z}_{\mathrm{odd}} := \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z} = \{i + \frac{1}{2} : i \in \mathbb{Z}\}$. It is more convenient for a symplectic flag to make a shift in the indices (either by an integer or by a half of an integer) such that $(\Lambda_i)^{\angle} = \Lambda_{-i+1}$ for any i in $\mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}_{\mathrm{odd}}$. In the sequel the subspaces in symplectic flags will be enumerated according to this rule.

Assume that $V = \bigoplus_{i \in A} V_i$, where A is either $\mathbb{Z}$ or $\mathbb{Z}_{\mathrm{odd}}$ is a graded space. We say that this grading is *symplectic* (or V is a *graded symplectic space*) if the corresponding flag

$$(7.1) \qquad\qquad \{V^i\}_{i \in \mathbb{Z} \text{ or } \frac{1}{2}\mathbb{Z}_{\mathrm{odd}}}, \text{ where } V^i = \bigoplus_{j \geq i} V_j,$$

is symplectic and also the subspaces V_i and V_j are skew orthogonal for all pairs (i, j) with $i + j \neq 0$. As in the previous subsection, the notion of indecomposibility plays the crucial role in the classification. Let δ_1 and δ_2 be degree -1 endomorphisms of the graded symplectic spaces V_1 and V_2, respectively, belonging to the corresponding symplectic algebras. The direct sum $V_1 \oplus V_2$ is equipped with the natural symplectic grading such that its ith component is the direct sum of ith components of V_1 and V_2 and the symplectic form on $V_1 \oplus V_2$ is defined as follows: the restriction of this form to V_i coincides with the symplectic form on V_i for each $i = 1, 2$ and the spaces V_1

and V_2 are skew-orthogonal in $V_1 \oplus V_2$. The direct sum $\delta_1 \oplus \delta_2$ is the degree -1 endomorphism of $V_1 \oplus V_2$ (belonging to $\mathfrak{sp}(V_1 \oplus V_2)$) such that the restriction of it to V_i is equal to δ_i for each $i = 1, 2$. A degree -1 endomorphism $\delta \in \mathfrak{sp}(V)$ of a graded symplectic space V is called *symplectically indecomposable* if it cannot be represented as a direct sum of two degree -1 endomorphisms acting on nonzero graded spaces and belonging to the corresponding symplectic algebras.

Remark 7.2. Note that we can add two graded symplectic spaces and two degree -1 endomorphisms on them if and only if either the gradings on both spaces are indexed by $\mathbb{Z}$ or the gradings on both spaces are indexed by $\frac{1}{2}\mathbb{Z}_{\mathrm{odd}}$.$\square$

Below we list two types of symplectically indecomposable degree -1 endomorphism:

(D1) Given a nonnegative $s \in \frac{1}{2}\mathbb{Z}$, and an integer l such that $0 \le l \le 2[s]$ let $V_{s;l}^{\mathfrak{sp}}$ be a linear symplectic space with a basis

$$(7.2) \qquad \{e_{s-l}, \ldots, e_s, f_{-s}, \ldots, f_{l-s}\}$$

such that $\sigma(e_i, e_j) = \sigma(f_i, f_j) = 0$, $\sigma(e_i, f_{-i}) = (-1)^{s-i}$ and $\sigma(e_i, f_j) = 0$ if $j \ne -i$. Define the grading on $V_{s;l}^{\mathfrak{sp}}$ such that the ith component equal to the span of all vectors with index i appearing in the tuple (7.10). It is a symplectic grading. Then denote by $\delta_{s;l}^{\mathfrak{sp}}$ a degree -1 endomorphism of $V_{s;l}^{\mathfrak{sp}}$ from the symplectic algebra such that $\delta_{s;l}^{\mathfrak{sp}}(e_i) = e_{i-1}$ for $s - l + 1 \le i \le s$, $\delta_{s;l}^{\mathfrak{sp}}(e_{s-l}) = 0$, $\delta_{s;l}^{\mathfrak{sp}}(f_i) = f_{i-1}$ for $-s + 1 \le i \le l - s$, and $\delta_{s;l}^{\mathfrak{sp}}(f_{-s}) = 0$.

(D2) Given a positive $m \in \frac{1}{2}Z_{\mathrm{odd}}$ let

$$\mathcal{L}_m^{\mathfrak{sp}} = \bigoplus_{\substack{-m \le i \le m, \\ i \in \frac{1}{2}\mathbb{Z}_{\mathrm{odd}}}} E_i$$

be a symplectic graded spaces such that $\dim E_i = 1$ for every admissible i and let $\tau_m^{\mathfrak{sp}}$ be a degree -1 endomorphism of $\mathcal{L}_m$ from the symplectic algebra which sends E_i onto E_{i-1} for every admissible i, except $i = -m$, and $\tau_m^{\mathfrak{sp}}(E_{-m}) = 0$. In the case $\mathbb{K} = \mathbb{R}$ we also assume that $\sigma(\tau_m^{\mathfrak{sp}}(e), e) \ge 0$ for all $e \in E_{\frac{1}{2}}$.

Remark 7.3. Note that the symbol $\tau_m^{\mathfrak{sp}}$ from (D2) corresponds to one (weighted) Jordan block acting on a symplectic space, while the symbol $\delta_{s;l}^{\mathfrak{sp}}$ from (D1)corresponds to two weighted Jordan blocks. Since each of these two blocks acts on a Lagrangian subspace of $V_{s;l}^{\mathfrak{sp}}$, we cannot decompose the symbol $\delta_{s;l}^{\mathfrak{sp}}$ symplectically. This is the main difference between the cases of $Sp(W)$ and of $GL(W)$.$\square$

By Remark 7.2 we cannot take the direct sum of $\delta_{s;l}^{\mathfrak{sp}}$ with $s \in \mathbb{Z}$ and $\tau_m^{\mathfrak{sp}}$, for example. The following theorem gives the classification of all degree -1 endomorphisms from the symplectic algebra of a graded symplectic space V and consequently the classification of all symbols of curves (both of unparametrized and parametrized) of symplectic flags of V with respect to $Sp(V)$ and $CSp(V)$:

Theorem 7.2. *Assume that V is a graded symplectic space.*

(1) *If the grading on V is indexed by $\mathbb{Z}$ then a degree -1 endomorphism from $\mathfrak{sp}(V)$ is conjugate (by a symplectic transformation) to the direct sum of endomorphisms of type $\delta_{s;l}^{\mathfrak{sp}}$, where s is a nonnegative integer and $0 \le l \le 2s$. Moreover, for each pair of integers (s, l) with $0 \le l \le 2s$ the number of appearances of $\delta_{s;l}^{\mathfrak{sp}}$ in this direct sum is an invariant of the conjugacy class.*

(2) *If the grading on V is indexed by $\frac{1}{2}\mathbb{Z}_{\text{odd}}$ and $\mathbb{K} = \mathbb{C}$, then a degree -1 endomorphism from $\mathfrak{sp}(V)$ is conjugate (by a symplectic transformation) to the direct sum of endomorphisms of type $\delta_{s;l}^{\mathfrak{sp}}$ with $s \in \frac{1}{2}\mathbb{Z}_{\text{odd}}$ and $0 \leq l \leq 2s - 1$, and of type $\tau_m^{\mathfrak{sp}}$ with positive $m \in \frac{1}{2}\mathbb{Z}_{\text{odd}}$. Moreover, for each pair $(s, l) \in \frac{1}{2}\mathbb{Z}_{\text{odd}} \times \mathbb{Z}$ with $0 \leq l \leq 2s-1$ and a positive $m \in \frac{1}{2}\mathbb{Z}_{\text{odd}}$ the numbers of appearances of $\delta_{s;l}^{\mathfrak{sp}}$ and $\tau_m^{\mathfrak{sp}}$ in this direct sum are invariants of the conjugacy class.*

(3) *If the grading on V is indexed by $\frac{1}{2}\mathbb{Z}_{\text{odd}}$ and $\mathbb{K} = \mathbb{R}$, then a degree -1 endomorphism from $\mathfrak{sp}(V)$ is conjugate to the direct sum of endomorphisms of type $\delta_{s;l}^{\mathfrak{sp}}$, $\tau_m^{\mathfrak{sp}}$, and $-\tau_m^{\mathfrak{sp}}$. Moreover, for each pair $(s, l) \in \frac{1}{2}\mathbb{Z}_{\text{odd}} \times \mathbb{Z}$ with $0 \leq l \leq 2s - 1$ and a positive $m \in \frac{1}{2}\mathbb{Z}_{\text{odd}}$ the numbers of appearances of $\delta_{s;l}^{\mathfrak{sp}}$, $\tau_m^{\mathfrak{sp}}$, and $-\tau_m^{\mathfrak{sp}}$ in this direct sum are invariants of the conjugacy class.*

Theorem 7.2 shows that, up to a conjugation, the only symplectically indecomposable endomorphisms are the endomorphisms of the type $\delta_{s;l}^{\mathfrak{sp}}$ and $\tau_m^{\mathfrak{sp}}$ for $\mathbb{K} = \mathbb{C}$ and of the type $\delta_{s;l}^{\mathfrak{sp}}$, $\tau_m^{\mathfrak{sp}}$, and $-\tau_m^{\mathfrak{sp}}$ in the case $\mathbb{K} = \mathbb{R}$.

Proof. Let δ be a degree -1 endomorphism of V belonging to $\mathfrak{sp}(V)$. Assume that $\delta^{l+1} = 0$ and $\delta^l \neq 0$ for some $l \geq 0$. Let $\mathcal{C}$ be a complement to $\ker \delta^l$ in V, i.e.

$$(7.3) \qquad\qquad V = \mathcal{C} \oplus \ker \delta^l.$$

Define a bilinear form $b : \mathcal{C} \times \mathcal{C} \mapsto \mathbb{K}$ by

$$(7.4) \qquad\qquad b(u_1, u_2) := \sigma(u_1, \delta^l u_2).$$

Since $\delta \in \mathfrak{sp}(V)$ and σ is skew-symmetric, the form b is symmetric if l is odd and skew-symmetric if l is even. As a matter of fact the form b can be considered as a bilinear form on $V / \ker \delta^l$.

Lemma 7.1. *The form b is non-degenerate.*

Proof. Assume that there exists $u_1 \in \mathcal{C}$ such that $b(u_1, u_2) = 0$ for any $u_2 \in \mathcal{C}$. In other words, $\sigma(u_1, \delta^l u_2) = 0$ for any $u_2 \in \mathcal{C}$. This together with splitting (7.3) implies that $\sigma(u_1, \delta^l v) = 0$ for any $v \in V$. Note that from the fact that $\delta \in \mathfrak{sp}(V)$ it follows that $\sigma(u_1, \delta^l v) = (-1)^l \sigma(\delta^l u_1, v)$, therefore $\sigma(\delta^l u_1, v) = 0$ for any $v \in V$. Since σ is non-degenerate, we get $\delta^l u_1 = 0$, i.e. $u_1 \in \ker \delta^l$. As a consequence of this, our assumption that $u_1 \in \mathcal{C}$, and splitting (7.3) we get that $u_1 = 0$. This completes the proof of the lemma. $\qquad \square$

Further from the splitting (7.3) it follows that $\delta^i \mathcal{C} \cap \delta^j \mathcal{C} = 0$ for $i > j$. Set

$$(7.5) \qquad\qquad Y_l := \mathcal{C} \oplus \delta \mathcal{C} \oplus \ldots \oplus \delta^l \mathcal{C}.$$

Lemma 7.2. *The restriction of the symplectic form σ to the subspace Y_l is non-degenerate.*

Proof. From Lemma 7.1 it follows that for any i the bilinear form $(u_1, u_2) \mapsto \sigma(\delta^i u_1, \delta^{l-i} u_2)$ is non-degenerate. Also the condition $\delta^{l+1} = 0$ implies that $\sigma(\delta^i u_1, \delta^j u_2) = 0$ for $i + j \geq l$. This implies that the matrix of the form $\sigma|_{Y_l}$ with respect to any basis of Y_l is block-triangle with respect to non-principal diagonal and each non-principal diagonal block is nonsingular. This completes the proof of the lemma. $\qquad \square$

We can always choose the space $\mathcal{C}$ in (7.3) such that $\mathcal{C}$ is a direct sum of homogeneous spaces (i.e. a direct sum of subspaces of V_i). By the previous lemma, $V = Y_l \oplus Y_l^{\angle}$, where $Y_l^{\angle}$ is the skew-symmetric complement of Y_l. By constructions $Y_l^{\angle} \subset \ker \delta^l$. Also it is easy to show that $Y_l^{\angle}$

inherits the symplectic grading from V, $Y_l^{\angle} = \bigoplus_{i \in \mathbb{Z}} Y_l^{\angle} \cap V_i$. Repeat the same procedure for $Y_l^{\angle}$ instead of V, then, if necessary, repeat it again. In this way one gets the unique skew-orthogonal splitting of V into the direct sum of invariant subspaces of δ,

$$(7.6) \qquad V = \bigoplus_{i=1}^{d} Y_{l_i}, \quad l = l_1 > l_2 > \ldots > l_d,$$

such that $\delta^{l_i+1}|_{Y_{l_i}} = 0$, $\delta^{l_i}|_{Y_{l_i}} \neq 0$, $Y_{l_i} = \mathcal{C}_i \oplus \delta \mathcal{C}_i \oplus \ldots \oplus \delta^{l_i} \mathcal{C}_i$ for a complement $\mathcal{C}_i$ to $\ker \delta^{l_i}|_{Y_{l_i}}$ in Y_{l_i}, and each Y_{l_i} is a graded symplectic space with the grading inherited from V.

From the splitting (7.6) it follows that to prove our theorem it is sufficient to restrict ourself to the case when $V = Y_l$, where Y_l is as in (7.5). Assume that s is the maximal nontrivial degree in the grading of V (i.e $V_s \neq 0$ and $V_i = 0$ for $i > s$). Let Z_s be the space of all vectors of degree s in V. Then by constructions Z_s is transversal to $\ker \delta^l$. Let

$$\mathcal{X} = Z_s \oplus \delta Z_s \oplus \ldots \oplus \delta^l Z_s$$

and $\widetilde{Z}_s$ be the image of Z_s in $V/\ker \delta^l$ under the canonical projection on the quotient space. In the sequel we will look on the form b as on a bilinear form on $V/\ker \delta^l$.

Lemma 7.3. *The space $\mathcal{X} \cap \mathcal{X}^{\angle}$ is equal to 0 or $\mathcal{X}$.*

Proof. By the same arguments as in the proof of Lemma 7.2 the statement of the present lemma is equivalent to the following statement: the restriction $b|_{\widetilde{Z}_s}$ of the form b on the space $\widetilde{Z}_s$ is either equal to 0 identically or non-degenerate. Set $K_1 = \ker b|_{\widetilde{Z}_s}$. Assume by contradiction that K_1 is a nonzero proper subspace of Z_s. Let $\{V^j\}_{j \in \mathbb{Z} \text{ or } \frac{1}{2}\mathbb{Z}_{\text{odd}}}$ be the flag as in (7.1). From the assumption that $K_1 \neq \widetilde{Z}_s$ it follows that $\delta Z_s \not\subseteq (V^{s-l+2})^{\angle}$. Hence

$$(7.7) \qquad V^{s-1} \not\subseteq (V^{s-l+2})^{\angle}.$$

Since the flag $\{V^j\}_{j \in \mathbb{Z}}$ is symplectic, the space $(V^{s-l+2})^{\angle}$ has to be equal to one of the subspaces of the flag $\{V^j\}_{j \in \mathbb{Z}}$. This together with the assumption that s is the maximal degree in the grading of V and relation (7.7) implies that

$$(7.8) \qquad (V^{s-l+2})^{\angle} = V_s = Z_s.$$

Now let $K_1^{\perp} = \{v \in V/\ker \delta^l : b(v,u) = 0 \quad \forall u \in K_1\}$. By constructions, $\widetilde{Z}_s \subseteq K_1^{\perp}$. The assumption $K_1 \neq 0$ implies that $K_1^{\perp}$ is a proper subspace of $V/\ker \delta^l$. Therefore there exist elements of degree less than s in V that are transversal to $\ker \delta^l$. Consequently the space V^{s-l+1} is a proper subspace of V and $V^{s-l+2} \subsetneq V^{s-l+1}$ This yields in turn that that $(V^{s-l+1})^{\angle}$ is a nonzero proper subspace of $(V^{s-l+2})^{\angle} = V_s$, which contradicts the fact that the flag $\{V^j\}_{j \in \mathbb{Z}}$ is symplectic and that s is the maximal degree in the grading of V. The proof of the lemma is completed. $\square$

Now consider separately the cases $\mathcal{X} \cap \mathcal{X}^{\angle} = 0$ and $\mathcal{X} \cap \mathcal{X}^{\angle} = \mathcal{X}$.

Lemma 7.4. *If $\mathcal{X} \cap \mathcal{X}^{\angle} = 0$, then $V = \mathcal{X}$.*

Proof. $V = \mathcal{X} \oplus \mathcal{X}^{\angle}$. Let us prove that $\mathcal{X}^{\angle} = 0$ or, equivalently, $V = \mathcal{X}$. Indeed, assume by contradiction that $\mathcal{X}^{\angle} \neq 0$. Then $\mathcal{X}^{\angle}$ is a graded symplectic spaces with the symplectic forms and the gradings inherited from V such that the maximal degree in the grading is less than s. This together with the assumption that $V = Y_l$, where Y_l is as in (7.5), implies that in the grading of

V there are nonzero vectors of degree less than $s - l + 1$. This contradicts relation (7.8) that must hold in the considered case. □

Lemma 7.5. *Assume that $\mathcal{X} \cap \mathcal{X}^{\angle} = 0$. Then the following three statements hold:*

(1) *If l is even, then the endomorphism δ is conjugate to the direct sum of the endomorphisms $\delta_{s,2s}^{\mathfrak{sp}}$ with an integer s repeated $\frac{1}{2} \dim Z_s$ times;*

(2) *If l is odd and $\mathbb{K} = \mathbb{C}$, then the endomorphism δ is conjugate to the direct sum of the endomorphisms $\tau_s^{\mathfrak{sp}}$ repeated $\dim Z_s$ times;*

(3) *If l is odd and $\mathbb{K} = \mathbb{R}$, then the endomorphism δ is conjugate to the direct sum of the endomorphisms $\tau_s^{\mathfrak{sp}}$ and $-\tau_s^{\mathfrak{sp}}$ such that the numbers of appearances of $\tau_{s/2}$ and $-\tau_{s/2}$ are equal to the positive and the negative indices of the form b, respectively.*

Proof. In the considered case as a subspace $\mathcal{C}$ in (7.3) one can take Z_s. As was already mentioned before, the form b (defined on Z_s) is non-degenerate symmetric if l is odd and non-degenerate skew-symmetric if l is even. If l is even we can choose a Darboux (symplectic) bases in Z_s with respect to the form b, i.e. a basis $\{\varepsilon_j, \nu_j\}_{j=1}^{\frac{1}{2} \dim Z_s}$ such that the form b is nonzero (and equal to ± 1) only for the pairs (ε_j, ν_j) and (ν_j, ε_j). Then for each $1 \le j \le \frac{1}{2} \dim Z_s$ the restriction of the endomorphism δ to the minimal invariant subspaces of δ containing ε_j and ν_j is conjugate to $\delta_{s;2s}^{\mathfrak{sp}}$, which proves item (1) of the lemma. If l is odd we choose a basis of Z_s for which the quadratic form corresponding to the form b is diagonal. Then for each vector of this basis the restriction of δ to the minimal invariant subspaces of δ containing this vector is conjugate to $\tau_s^{\mathfrak{sp}}$, which proves items (2) and (3) of the lemma. □

Now consider the case $\mathcal{X} \cap \mathcal{X}^{\angle} = \mathcal{X}$. In this case $\widetilde{Z}_s \subset \widetilde{Z}_s^{\perp}$. The grading on V induces the natural grading on $V/\ker \delta^l$. Take a subspace $\widetilde{K}_2$ consisting of the homogeneous elements of the minimal degree s_1 in $V/\ker \delta^l$. By constructions $s_1 < s$. Let us prove that

$$(7.9) \qquad\qquad V/\ker \delta^l = \widetilde{Z}_s^{\perp} \oplus \widetilde{K}_2.$$

Take a subspace K_2 which is a representative of $\widetilde{K}_2$ in V and consists of homogeneous vectors (of degree s_1). From minimality of s_1 it follows that $V_{s_1-l+1} = \delta^l K_2$ and $V_i = 0$ for $i > s_1 - l + 1$. From the maximality of s it follows that $Z_s = (V^{s_1-l+2})^{\angle}$. This implies that $\dim \widetilde{K}_2 = \dim \widetilde{Z}_s$ and that $\widetilde{K}_2 \cap \widetilde{Z}_s^{\perp} = 0$, which in turn yields (7.9).

Now let $\mathcal{Y} = K_2 \oplus \delta K_2 \oplus \ldots \oplus \delta^l K_2$. By constructions, the restriction of the endomorphism δ to $\mathcal{X} \oplus \mathcal{Y}$ is conjugate to the direct sum of endomorphisms $\delta_{s;l}^{\mathfrak{sp}}$ repeated $\dim Z_s$ times Further consider the space $(\mathcal{X} \oplus \mathcal{Y})^{\angle}$. This space inherits the grading from V, $(\mathcal{X} \oplus \mathcal{Y})^{\angle} = \bigoplus_{i \in \mathbb{Z}} (\mathcal{X} \oplus \mathcal{Y})^{\angle} \cap V_i$, with the maximal nontrivial grading less than s. Repeat the same procedure for $(\mathcal{X} \oplus \mathcal{Y})^{\angle}$ instead of V, then, if necessary, repeat it again. In this way we decompose the endomorphism δ in the case of $V = Y_l$ to the direct sum of symbols of the type $\delta_{s;l}^{\mathfrak{sp}}$ and $\tau_m^{\mathfrak{sp}}$ for $\mathbb{K} = \mathbb{C}$ or $\delta_{s;l}^{\mathfrak{sp}}$, $\tau_m^{\mathfrak{sp}}$, and $-\tau_m^{\mathfrak{sp}}$ in the case $\mathbb{K} = \mathbb{R}$. This together with the decomposition (7.6) completes the proof of Theorem (7.2). □

Remark 7.4. In the works [41, 42] of the second author with C. Li parameterized curves in Lagrangian Grassmannian (over $\mathbb{R}$) satisfying so-called condition (G) were considered. In the present terminology condition (G) of [41, 42] is equivalent to the condition that the symbol of the parameterized curve of symplectic flag is conjugate to the direct sum of endomorphisms of type τ_m and $-\tau_m$. The Young diagram which were assigned there to a curve in Lagrangian Grassmannian

can be described as follows: for any $p \in \mathbb{N}$ the number of rows of length p in it is equal to the number of appearances of endomorphisms $\tau^{\mathfrak{sp}}_{\frac{2p-1}{2}}$ and $-\tau^{\mathfrak{sp}}_{\frac{2p-1}{2}}$ in this direct sum. We calculate the universal prolongation of such symbols in subsection 8.3.6 below, which together with Theorem 4.1 gives more conceptual point of view on the constructions of papers [41, 42] and generalize them to more general classes of curves.

7.3. Orthogonal case.
The classification of symbols in this case is very similar to the symplectic case. As in the symplectic case we make a shift in the indices (either by an integer or by a half of an integer) such that $(\Lambda_i)^\perp = \Lambda_{-i+1}$ for any i in $\mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}_{\mathrm{odd}}$, where $L^\perp$ denotes the orthogonal complement of a subspace L with respect to the non-degenerate symmetric form Q. Further, by complete analogy with the symplectic space we can define graded orthogonal spaces and orthogonally indecomposable degree -1 endomorphisms.

By analogy with (D1) and (D2) of the previous subsection, there are the following two types of orthogonally indecomposable degree -1 endomomorphisms:

(E1) Given a positive $s \in \frac{1}{2}\mathbb{Z}$, and an integer l such that $0 \le l \le 2(s + \{s\}) - 1$ let $V^{\mathfrak{so}}_{s;l}$ be a linear space equipped with a non-degenerate symmetric form and with a basis

$$(7.10) \qquad \{e_{s-l}, \dots, e_s, f_{-s}, \dots, f_{l-s}\}$$

such that $Q(e_i, e_j) = Q(f_i, f_j) = 0$, $\sigma(e_i, f_j) = (-1)^{s-i}$, if $j = -i$, and $\sigma(e_i, f_j) = 0$ if $j \ne -i$. Define the grading on $V^{\mathfrak{so}}_{s;l}$ such that the ith component equal to the span of all vectors with index i appearing in the tuple (7.10). It is an orthogonal grading. Then denote by $\delta^{\mathfrak{so}}_{s;l}$ a degree -1 endomorphism of $V^{\mathfrak{so}}_{s;l}$ from the orthogonal algebra such that $\delta^{\mathfrak{so}}_{s;l}(e_i) = e_{i-1}$ for $s - l + 1 \le i \le s$, $\delta^{\mathfrak{so}}_{s;l}(e_{s-l}) = 0$, $\delta^{\mathfrak{so}}_{s;l}(f_i) = f_{i-1}$ for $-s + 1 \le i \le l - s$, and $\delta^{\mathfrak{so}}_{s;l}(f_{-s}) = 0$.

(E2) Given a nonnegative integer m let $\mathcal{L}^{\mathfrak{so}}_m = \bigoplus_{i=-m}^m E_i$ be an orthogonal graded spaces such that $\dim E_i = 1$ for every admissible i and let $\tau^{\mathfrak{so}}_m$ be a degree -1 endomorphism of $\mathcal{L}_m$ from the orthogonal algebra which sends E_i onto E_{i-1} for every admissible i, except $i = -m$, and $\tau^{\mathfrak{so}}_m(E_{-m}) = 0$. In the case $\mathbb{K} = \mathbb{R}$ we also assume that $Q(\tau^{\mathfrak{so}}_m(e), e) \ge 0$ for all $e \in E_1$.

Remark 7.5. Note that the symbol $\tau^{\mathfrak{so}}_m$ from (E2) corresponds to one (weighted) Jordan block acting on a space equipped with a non-degenerate symmetric form, while the symbol $\delta^{\mathfrak{so}}_{s;l}$ from (E1) corresponds to two weighted Jordan blocks. Since each of these two blocks acts on a maximal isotropic subspace of $V^{\mathfrak{so}}_{s;l}$, we cannot decompose the symbol $\delta^{\mathfrak{so}}_{s;l}$ orthogonally. This is the main difference between the cases of $O(W)$ and of $GL(W)$.$\square$

The following theorem gives the classification of all degree -1 endomorphisms from the orthogonal algebra of a graded orthogonal space V and consequently the classification of all symbols of curves (both of unparametrized and parametrized) of orthogonal flags of V with respect to $O(V)$ and $CO(V)$:

Theorem 7.3. *Assume that V is a graded orthogonal space.*

(1) *If the grading on V is indexed by $\frac{1}{2}\mathbb{Z}_{\mathrm{odd}}$ then a degree -1 endomorphism from $\mathfrak{so}(V)$ is conjugate (by an orthogonal transformation) to the direct sum of endomorphisms of type $\delta^{\mathfrak{so}}_{s;l}$, where $s \in \frac{1}{2}\mathbb{Z}_{\mathrm{odd}}$ and $0 \le l \le 2s$. Moreover, for each pair of integers $(s, l) \in \frac{1}{2}\mathbb{Z}_{\mathrm{odd}} \times \mathbb{Z}$ with $0 \le l \le 2s$ the number of appearances of $\delta^{\mathfrak{so}}_{s;l}$ in this direct sum is an invariant of the conjugacy class.*

(2) *If the grading on V is indexed by Z and $\mathbb{K} = \mathbb{C}$, then a degree -1 endomorphism from $\mathfrak{so}(V)$ is conjugate (by an orthonormal transformation) to the direct sum of endomorphisms of type $\delta_{s;l}^{\mathfrak{so}}$ with a positive integer s and $0 \le l \le 2s-1$, and of type $\tau_m^{\mathfrak{so}}$ with a nonnegative integer m. Moreover, for each pair of integers (s,l) with $0 \le l \le 2s-1$ a nonnegative integer m the numbers of appearances of $\delta_{s;l}^{\mathfrak{so}}$ and $\tau_m^{\mathfrak{so}}$ in this direct sum are invariants of the conjugacy class.*

(3) *If the grading on V is indexed by $\frac{1}{2}\mathbb{Z}_{\mathrm{odd}}$ and $\mathbb{K} = \mathbb{R}$, then a degree -1 endomorphism from $\mathfrak{so}(V)$ is conjugate to the direct sum of endomorphisms of type $\delta_{s;l}^{\mathfrak{so}}$, $\tau_m^{\mathfrak{so}}$, and $-\tau_m^{\mathfrak{so}}$. Moreover, for each pair of integers (s,l) with $0 \le l \le 2s-1$ and a nonnegarive integer m the numbers of appearances of $\delta_{s;l}^{\mathfrak{so}}$, $\tau_m^{\mathfrak{so}}$, and $-\tau_m^{\mathfrak{so}}$ in this direct sum are invariants of the conjugacy class.*

Theorem 7.3 shows that, up to a conjugation, the only orthogonally indecomposable endomorphisms are the endomorphisms of the type $\delta_{s;l}^{\mathfrak{so}}$ and τ_m for $\mathbb{K} = \mathbb{C}$ and of the type $\delta_{s;l}^{\mathfrak{so}}$, $\tau_m^{\mathfrak{so}}$, and $-\tau_m^{\mathfrak{so}}$ in the case $\mathbb{K} = \mathbb{R}$.

The proof of Theorem 7.3 is identical to the proof of Theorem 7.2. The only difference is that in the present case the form b, defined by (7.4), is symmetric if l is even and skew-symmetric if l is odd. Therefore Lemma 7.5 should be modified appropriately.

Remark 7.6. This remark is about a possible relation of the problem of classification of symbols to the theory of quiver representations ([21], [8]). In the case $G = GL(W)$(or $SL(W)$) there is an obvious one-to-one correspondence between the set of symbols (obtained in Theorem 7.1) and the set of indecomposable representations of the quivers with underlying undirected graph equal to the Dynkin diagram of type A_ℓ (if one does not take into account possible shift in the range of indices in the graded space). It would be interesting to link the obtained classification of indecomposable symplectic and orthogonal symbols with representations of quivers with the corresponding Dynkin diagrams as underlying indirect graphs. $\square$

8. Computation of algebraic prolongation of symbols for classical groups

8.1. **Decomposition of the universal prolongation algebra.** First we point out some general properties of the universal algebraic prolongation in the case when $\mathfrak{g}$ is a graded reductive Lie algebra. Let $\mathfrak{m}$ be a line in $\mathfrak{g}_{-1}$ and $\mathfrak{u}(\mathfrak{m})$ be the universal algebraic prolongation of $\mathfrak{m}$, as defined in subsection 4.1. Take $\delta \in \mathfrak{m}$. According to Jacobson-Morozov theorem [26, Ch.III, Th. 17] (see also [39] for complex graded Lie algebras and [9] for real graded case), we can complete δ by elements H and Y of degree 0 and 1 respectively (in $\mathfrak{g}$) to the standard basis of a $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$, i.e such that

$$(8.1) \qquad\qquad [H,\delta] = 2\delta, \ [H,Y] = -2Y, \ [\delta,Y] = H.$$

Let $\mathfrak{n}_{\max}(\mathfrak{m})$ be the largest ideal in $\mathfrak{u}(\mathfrak{m})$ concentrated in the non-negative degree (i.e., $\mathfrak{n}_{\max}(\mathfrak{m}) \subset \sum_{i \ge 0} \mathfrak{u}_i$). Such ideal exists since the sum of any two ideals concentrated in the non-negative degree will also be an ideal of this type. It is also clear that $\mathfrak{n}_{\max}(\mathfrak{m})$ is graded, i.e. $\mathfrak{n}_{\max}(\mathfrak{m}) = \sum_i (\mathfrak{n}_{\max}(\mathfrak{m}))_i$, where $(\mathfrak{n}_{\max}(\mathfrak{m}))_i = \mathfrak{n}_{\max}(\mathfrak{m}) \cap \mathfrak{g}_i$.

On the one hand, since $[\delta, \mathfrak{n}_{\max}(\mathfrak{m})] \subset \mathfrak{n}_{\max}(\mathfrak{m})$ and relations (8.1) hold, we get that $\mathfrak{sl}_2 \cap \mathfrak{n}_{\max}(\mathfrak{m}) = 0$, which implies that

$$(8.2) \qquad\qquad \dim \mathfrak{u}(\mathfrak{m})/\mathfrak{n}_{\max}(\mathfrak{m}) \ge 3.$$

On the other hand, under the identification of the algebra $\mathfrak{u}(\mathfrak{m})$ with the algebra of infinitesimal symmetries of a flat curve $F_{\mathfrak{m}}$ with the symbol $\mathfrak{m}$ (see subsection 4.1), to any element of $\mathfrak{u}(\mathfrak{m})$ we

can assign the vector field on the curve $F_{\mathfrak{m}}$. Consider the subspace $\mathfrak{n}_{ne}(\mathfrak{m})$ of $\mathfrak{u}(\mathfrak{m})$, consisting of all elements of $\mathfrak{u}(\mathfrak{m})$ for which the corresponding vector fields on $F_{\mathfrak{m}}$ are identically zero. Clearly, $\mathfrak{n}_{ne}(\mathfrak{m})$ is an ideal of $\mathfrak{u}(\mathfrak{m})$. This ideal is called the *non-effectiveness ideal of* $\mathfrak{u}(\mathfrak{m})$. Then the quotient algebra $\mathfrak{u}(\mathfrak{m})/\mathfrak{n}_{ne}(\mathfrak{m})$ can be realized as a finite-dimensional Lie algebra of vector fields on a curve. From a classical result by Sophus Lie result it follows that

$$(8.3) \qquad\qquad\qquad \dim \mathfrak{u}(\mathfrak{m})/\mathfrak{n}_{ne}(\mathfrak{m}) \leq 3$$

(see the original proof in [27] , its translation and commentary in [1], and a self-contained proof in the recent paper [17]). Since $\mathfrak{n}_{ne}(\mathfrak{m}) \subset \mathfrak{n}_{\max}(\mathfrak{m})$ we get from (8.2) and (8.3) that $\mathfrak{n}_{\max}(\mathfrak{m}) = \mathfrak{n}_{ne}(\mathfrak{m})$ and that $\dim \mathfrak{u}(\mathfrak{m})/\mathfrak{n}_{\max}(\mathfrak{m}) = 3$. The latter implies the following

Proposition 8.1. $\mathfrak{u}(\mathfrak{m})$ *is a semidirect sum of the constructed embedding of* $\mathfrak{sl}_2$ *into* $\mathfrak{g}$ *and* $\mathfrak{n}_{\max}(\mathfrak{m})$ *$(= \mathfrak{n}_{ne}(\mathfrak{m}))$. In particular,* $\mathfrak{n}_{\max}$ *is an* $\mathfrak{sl}_2$-*module.*

The latter fact is very useful in the description of the universal algebraic prolongations of symbols of (unparametrized) curves of flags.

Remark 8.1. For parametrized curves the corresponding universal prolongation of a symbol $\delta \in \mathfrak{g}_{-1}$ is equal to a semidirect sum of $\mathbb{K}\delta$ and the non-effectiveness ideal $\mathfrak{n}_{ne}$.

8.2. **The case of** $G = GL(W)$ **$(SL(W))$.** We say that a graded space V which is also $\mathfrak{sl}_2$-module is a *nice* $\mathfrak{sl}_2$-*module*, if the corresponding embedding of $\mathfrak{sl}_2$ into $\mathfrak{gl}(V)$ is spanned by endomorphisms of degree -1, 0, and 1. Let V_1 and V_2 be two nice $\mathfrak{sl}_2$-modules. Then $\mathrm{Hom}(V_1, V_2) = V_2 \otimes V_1^*$ is the $\mathfrak{sl}_2$-module and a graded space in a natural way. Denote by $\mathfrak{n}(V_1, V_2)$ the maximal $\mathfrak{sl}_2$-submodule of $\mathrm{Hom}(V_1, V_2)$ concentrated in the non-negative degree part.

Assume that $\mathfrak{m} = \mathbb{R}\delta$. By Theorem 7.1 there exists a map $N : \{(r,s) \in \mathbb{Z} \times \mathbb{Z} : r \leq s < 0\} \to \mathbb{N} \cup \{\infty\}$ with finite support such that δ is conjugate to the endomorphism D_N, which is the direct sum of indecomposable symbols where δ_{rs} is repeated $N(r,s)$ times. The endomorphism D_n acts on the space

$$(8.4) \qquad\qquad\qquad \mathcal{V}_N = \bigoplus_{r \leq s < 0} V_{rs} \otimes \mathbb{K}^{N(r,s)}.$$

First, D_N can be extended to a subalgebra of $\mathfrak{gl}(\mathcal{V}_N)$ isomorphic to $\mathfrak{sl}_2$ which acts irreducibly on each V_{rs}. From (8.4) it follows that

$$\mathfrak{gl}(\mathcal{V}_N) = \mathcal{V}_N \otimes (\mathcal{V}_N)^* = \bigoplus_{\substack{r_1 \leq s_1 < 0, \\ r_2 \leq s_2 < 0}} \mathrm{Hom}(V_{r_1 s_1}, V_{r_2 s_2}) \otimes \mathrm{Hom}(\mathbb{K}^{N(r_1,s_1)}, \mathbb{K}^{N(r_2,s_2)}).$$

Second, by definition of $\mathfrak{n}(V_{r_1 s_1}, V_{r_2 s_2})$ we have that

$$(8.5) \qquad\qquad \mathfrak{p} = \bigoplus_{\substack{r_1 \leq s_1 < 0, \\ r_2 \leq s_2 < 0}} \mathfrak{n}(V_{r_1 s_1}, V_{r_2 s_2}) \otimes \mathrm{Hom}(\mathbb{K}^{N(r_1,s_1)}, \mathbb{K}^{N(r_2,s_2)})$$

is the maximal $\mathfrak{sl}_2$-module in $\mathfrak{gl}(\mathcal{V}_N)$ concentrated in the non-negative degree part of $\mathfrak{gl}(\mathcal{V}_N)$. Besides, $\mathfrak{p}$ is a subalgebra of $\mathfrak{gl}(\mathcal{V}_N)$. Indeed, $[\mathfrak{p}, \mathfrak{p}]$ is an $\mathfrak{sl}_2$-module in $\mathfrak{gl}(\mathcal{V}_N)$ concentrate in the non-negative degree, which implies that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}$, because $\mathfrak{p}$ is the maximal $\mathfrak{sl}_2$-module satisfying this property. Therefore $p = \mathfrak{n}_{\max}(\mathbb{K}D_n)$.

It remains to describe $\mathfrak{n}(V_{r_1,s_1}, V_{r_2,s_2})$ more explicitly. Set $l_i = s_i - r_i$, $i = 1, 2$. Note that $\mathfrak{sl}_2$-submodule $V_{r_2 s_2} \otimes (V_{r_1 s_1})^*$ is decomposed into the irreducible $\mathfrak{sl}_2$-modules as follows:

$$(8.6) \qquad \operatorname{Hom}(V_{r_1 s_1}, V_{r_2 s_2}) = V_{r_2 s_2} \otimes (V_{r_1 s_1})^* \cong \bigoplus_{i=0}^{\min\{l_1, l_2\}} \Pi_{l_1 + l_2 - 2i},$$

where Π_j denotes an irreducible $\mathfrak{sl}_2$-module of dimension $j + 1$ (see, for example, [20]). Moreover, the submodule $\Pi_{l_1 + l_2}$ of the largest dimension is generated by the elements of highest (or lowest) degree in $V_{r_2 s_2} \otimes (V_{r_1 s_1})^*$, which is equal to $s_2 - r_1$ ($r_2 - s_1$ respectively). The range of degrees for each next submodule in this decomposition is shrunk by 1 from both left and right sides, i.e. the submodule $\Pi_{l_1 + l_2 - 2i}$ has degrees varying from $r_2 - s_1 + i$ to $s_2 - r_1 - i$. The submodule $\mathfrak{n}(V_{r_1 s_1}, V_{r_2 s_2})$ is equal to the direct sum of submodules from the decomposition (8.6) for which all degrees are non-negative. Therefore,

$$(8.7) \qquad \mathfrak{n}(V_{r_1 s_1}, V_{r_2 s_2}) \cong \bigoplus_{i=\max\{0, s_1 - r_2\}}^{\min\{l_1, l_2, s_2 - r_1\}} \Pi_{l_1 + l_2 - 2i}.$$

In particular,

$$(8.8) \qquad \mathfrak{n}(V_{rs}, V_{rs}) \cong \Pi_0 \cong \mathbb{K} \operatorname{Id},$$

$$(8.9) \qquad \mathfrak{n}(V_{r_1 s_1}, V_{r_2 s_2}) = 0 \text{ if and only if } s_2 < s_1 \text{ or } r_2 < r_1.$$

Let us prove statement (8.9). Indeed by (8.7) $\mathfrak{n}(V_{r_1 s_1}, V_{r_2 s_2}) = 0$ if and only if one of the following three conditions holds:

(1) $s_1 - r_2 > s_1 - r_1 \equiv r_2 < r_1$;
(2) $s_1 - r_2 > s_2 - r_2 \equiv s_2 < s_1$;
(3) $s_1 - r_2 > s_2 - r_1$,

which proves the "if" part of (8.9). Further, if conditions (1) and (2) does not hold then the condition (3) does not hold as well, which proves the "only if" part.

Further, it is clear that

$$(8.10) \qquad \mathfrak{n}(V_{r_1, s_1} \otimes \mathbb{K}^{N_1}, V_{r_2, s_2} \otimes \mathbb{K}^{N_2}) \cong \mathfrak{n}(V_{r_1, s_1}, V_{r_2, s_2}) \otimes \operatorname{Hom}\left(\mathbb{K}^{N(r_1, s_1)}, \mathbb{K}^{N(r_2, s_2)}\right).$$

Combining (8.5), (8.8), (8.9), (8.10), and Proposition 8.1 we get the following

Theorem 8.1. *The universal prolongation algebra* $\mathfrak{u}(\mathbb{K}D_N)$ *of the symbol* $\mathbb{K}D_N$ *is equal to the semidirect sum of the constructed embedding of* $\mathfrak{sl}_2$ *into* $\mathfrak{gl}(V)$ *and*

$$\bigoplus_{\substack{s_1 \leq s_2, \, r_1 \leq r_2, \\ r_j \leq s_j < 0}} \mathfrak{n}(V_{r_1 s_1}, V_{r_2 s_2}) \otimes \operatorname{Hom}\left(\mathbb{K}^{N(r_1, s_1)}, \mathbb{K}^{N(r_2, s_2)}\right),$$

where $\mathfrak{n}(V_{r_1 s_1}, V_{r_2 s_2})$ *is as in* (8.7).

In particular, if $\delta = \delta_{rs}$ then by (8.8) one has $\mathfrak{n}_{\max}(\delta) = \mathbb{K} \operatorname{Id}$ and $\mathfrak{u}(\delta) \cong \mathfrak{gl}_2$. This can be applied to Example 6.1 above.

Remark 8.2. It is clear that in the case of $G = SL(W)$ the universal prolongation of a symbol consists of the traceless part of the universal prolongation of the same symbol for $G = GL(W)$.

8.3. **The case of** $G = Sp(W)$ $(CSp(W))$ **and** $G = O(W)$ $(CO(W))$**.** First note that in the case $G = CSp(W)$ $(G = CO(W))$ the universal algebraic prolongation of a symbol is equal to the direct sum of the universal algebraic prolongation of the same symbol for $G = Sp(W)$ $(G = O(W))$ with $\mathbb{K}$. Therefore it is sufficient to concentrate on the case $G = Sp(W)$ and $G = O(W)$. We will primary treat the symplectic case and briefly indicate what changes should be done for the orthogonal case. Let V be a graded symplectic space which is also a nice $\mathfrak{sl}_2$-module such that the corresponding embedding of $\mathfrak{sl}_2$ into $\mathfrak{gl}(V)$ belongs to $\mathfrak{sp}(V)$. In this case we will say that V is a *nice symplectic $\mathfrak{sl}_2$-module*. The symplectic algebra $\mathfrak{sp}(V)$ is a $\mathfrak{sl}_2$-module and a graded space in a natural way. Denote by $\mathfrak{l}^{\mathfrak{sp}}(V)$ the maximal $\mathfrak{sl}_2$-submodule of $\mathfrak{sp}(V)$ concentrated in the non-negative degree part. From the maximality assumption it follows that $\mathfrak{l}^{\mathfrak{sp}}(V)$ is a subalgebra of $\mathfrak{sp}(V)$.

Further, assume that $V = \bigoplus_{i=1}^{q} L_i$, where each L_i is a graded symplectic space and nice symplectic $\mathfrak{sl}_2$-modules (with all structures induced from V). The restriction $\sigma|_{L_i}$ to L_i of the symplectic form σ of V defines natural identification between L_i and its dual space L_i^*. Here σ denotes the symplectic form on V. Consider the following splitting of $\mathfrak{gl}(V)$:

$$(8.11) \qquad \mathfrak{gl}(V) = \bigoplus_{i,j=1}^{q} \operatorname{Hom}(L_i, L_j).$$

An endomorphism $A \in \mathfrak{gl}(V)$, having the decomposition $A = \sum_{i,j=1}^{q} A_{ij}$ with respect to the splitting (8.11), belongs to $\mathfrak{sp}(V)$ if and only if $A_{ii} \in \mathfrak{sp}(L_i)$ for all $1 \leq i \leq q$ and $A_{ij} = -A_{ji}^*$ for all $1 \leq i \neq j \leq q$ (here the dual linear map A_{ji}^* is considered as a map from L_i to L_j under the aforementioned identification $L_i \sim L_i^*$ and $L_j \sim L_j^*$). Therefore the map $A \mapsto \sum_{i=1}^{q} A_{ii} + \sum_{1 \leq i < j \leq q} A_{ij}$ defines the identification

$$(8.12) \qquad \mathfrak{sp}\left(\bigoplus_{i=1}^{q} L_i\right) \cong \bigoplus_{i=1}^{q} \mathfrak{sp}(L_i) \oplus \bigoplus_{1 \leq i < j \leq q} \operatorname{Hom}(L_i, L_j).$$

Moreover,

$$(8.13) \qquad \mathfrak{l}^{\mathfrak{sp}}\left(\bigoplus_{i=1}^{q} L_i\right) \cong \bigoplus_{i=1}^{q} \mathfrak{l}^{\mathfrak{sp}}(L_i) \oplus \bigoplus_{1 \leq i < j \leq q} \mathfrak{n}(L_i, L_j),$$

where $\mathfrak{n}(L_i, L_j)$ is as in the previous subsection.

Now take a symplectic symbol $\mathfrak{m} = \mathbb{K}\delta$. Let $V_{s;l}^{\mathfrak{sp}}$ and $\mathcal{L}_m^{\mathfrak{sp}}$ be graded symplectic spaces as in items (D1) and (D2) of subsection 3. According to Theorem 7.2 δ is conjugate to a direct sum of endomorphism of types $\delta_{s;l}^{\mathfrak{sp}}$, $\tau_m^{\mathfrak{sp}}$, and , in the case of $\mathbb{K} = \mathbb{R}$, also of type $-\tau_m^{\mathfrak{sp}}$. Therefore one can always assume that $V = \bigoplus_{i=1}^{q} L_i$, where each L_i is either equal to $V_{s;l}^{\mathfrak{sp}}$ or $\mathcal{L}_m^{\mathfrak{sp}}$. In the symplectic case the endomorphism δ can be extended to a subalgebra of $\mathfrak{sp}\left(\bigoplus_{i=1}^{q} L_i\right)$ isomorphic to $\mathfrak{sl}_2$ such that if $L_i = V_{s;l}^{\mathfrak{sp}}$ then $V_{s;l}^{\mathfrak{sp}}$ (with respect to the induced action) is the sum of two irreducible $\mathfrak{sl}_2$-submodules

$$(8.14) \qquad E_{s;l} = \operatorname{span}\{e_i\}_{s-l \leq i \leq s}, \quad F_{s;l} = \operatorname{span}\{f_i\}_{-s \leq i \leq l-s},$$

where e_i and f_i are as in (7.10), and if $L_i = \mathcal{L}_m^{\mathfrak{sp}}$, then $\mathcal{L}_m^{\mathfrak{sp}}$ (with respect to the induced action) is an irreducible $\mathfrak{sl}_2$-module. By analogy with the previous subsection *the universal algebraic prolongation* $\mathfrak{u}(\mathbb{R}\delta)$ *of the symbol* $\mathbb{R}\delta$ *is equal to the semidirect sum of the constructed embedding of* $\mathfrak{sl}_2$ *into* $\mathfrak{sp}(V)$ *and the algebra* $\mathfrak{l}^{\mathfrak{sp}}\left(\bigoplus_{i=1}^q L_i\right)$.

The orthogonal case is treated in completely the same way. We only need to replace everywhere from the beginning of this subsection the word "symplectic" by the word "orthogonal", the sign $\mathfrak{sp}$ by the sign $\mathfrak{so}$, and the symplectic form σ by a non-degenerate symmetric form Q.

By identification (8.13) and the analogous formula for the orthogonal case in order to compute $\mathfrak{l}^{\mathfrak{sp}}\left(\bigoplus_{i=1}^q L_i\right)$ $\left(\mathfrak{l}^{\mathfrak{so}}\left(\bigoplus_{i=1}^q L_i\right)\right.$ and, consequently, the universal prolongation $\mathfrak{u}(\mathbb{R}\delta)$ in $\mathfrak{sp}(V)$ $(\mathfrak{so}(V))$ it is sufficient to compute spaces $\mathfrak{l}^{\mathfrak{sp}}(V)$ $(\mathfrak{l}^{\mathfrak{spo}}(V))$ and $\mathrm{Hom}(V_1, V_2)$, where V, V_1, and V_2 are equal either to $V_{l;s}^{\mathfrak{sp}}$ or $\mathcal{L}_m^{\mathfrak{sp}}$ (either to $V_{l;s}^{\mathfrak{so}}$ or $\mathcal{L}_m^{\mathfrak{so}}$).

For this first denote by $\mathfrak{n}_{(r_1,s_1),(r_2,s_2)}$ the right-hand side of (8.7),

$$(8.15) \qquad \mathfrak{n}_{(r_1,s_1),(r_2,s_2)} = \bigoplus_{i=\max\{0,s_1-r_2\}}^{\min\{l_1,l_2,s_2-r_1\}} \Pi_{l_1+l_2-2i},$$

where $l_i = s_i - r_i$.

8.3.1. *Description of* $\mathfrak{l}^{\mathfrak{sp}}(V_{s;l}^{\mathfrak{sp}})$ *and* $\mathfrak{l}^{\mathfrak{so}}(V_{s;l}^{\mathfrak{so}})$.

First let us give a convenient description of the algebras $\mathfrak{sp}\left(V_{s;l}^{\mathfrak{sp}}\right)$ and $\mathfrak{so}\left(V_{s;l}^{\mathfrak{so}}\right)$. In the sequel we will use the same notation for $E_{s;l}$ and $F_{s;l}$ in the orthogonal case, as in (8.14). The symplectic form or the non-degenerate symmetric form define the following natural identifications: $E_{s;l} \cong (F_{s;l})^*$ and $F_{s;l} \cong (E_{s;l})^*$. Keeping in mind these identifications, an endomorphism $A \in \mathfrak{gl}(V)$, having decomposition $A = A_{11} + A_{22} + A_{12} + A_{21}$ with respect to the splitting

$$\mathfrak{gl}(V_{s;l}^{\mathfrak{sp}}) = \mathfrak{gl}(E_{s;l}) \oplus \mathfrak{gl}(F_{s;l}) \oplus \mathrm{Hom}(F_{s;l}, E_{s;l}) \oplus \mathrm{Hom}(E_{s;l}, F_{s;l}),$$

belongs to $\mathfrak{sp}\left(V_{s;l}^{\mathfrak{sp}}\right)$ if and only if $A_{12}^* = A_{12}$, $A_{21}^* = A_{21}$, and $A_{22} = -A_{11}^*$. It belongs to $\mathfrak{so}\left(V_{s;l}^{\mathfrak{so}}\right)$ if and only if $A_{12}^* = -A_{12}$, $A_{21}^* = -A_{21}$, and $A_{22} = -A_{11}^*$. Therefore the map $A \mapsto A_{11} + A_{22} + A_{12}$ defines the following identifications

$$(8.16) \qquad \mathfrak{sp}(V_{s;l}^{\mathfrak{sp}}) \cong \mathfrak{gl}(E_{s;l}) \oplus S^2(E_{s;l}) \oplus S^2(F_{s;l}),$$

$$(8.17) \qquad \mathfrak{so}(V_{s;l}^{\mathfrak{so}}) \cong \mathfrak{gl}(E_{s;l}) \oplus S^2(E_{s;l}) \oplus \wedge^2(F_{s;l}),$$

where $S^2(E_{s;l})$ and $S^2(F_{s;l})$ denote the symmetric square of $E_{s;l}$ and $F_{s;l}$, respectively, and $\wedge^2(E_{s;l})$ and $\wedge^2(F_{s;l})$ denote the skew-symmetric square of $E_{s;l}$ and $F_{s;l}$.

Here $S^2(E_{s;l})$ and $\wedge^2(E_{s;l})$ are subspaces of $\mathrm{Hom}(F_{s;l}, E_{s;l})$; $S^2(F_{s;l})$ and $\wedge^2(F_{s;l})$ are subspaces of $\mathrm{Hom}(E_{s;l}, F_{s;l})$. Keeping this in mind, we define $\mathfrak{l}_{s;l}^{\mathfrak{sp},1}$ $(\mathfrak{l}_{s;l}^{\mathfrak{so},1})$ as the maximal $\mathfrak{sl}_2$-submodule of $S^2(E_{s;l})$ $(\wedge^2(E_{s;l}))$ concentrated in the non-negative degree part of $S^2(E_{s;l})$ $(\wedge^2(E_{s;l}))$. Similarly, let $\mathfrak{l}_{s;l}^{\mathfrak{sp},2}$ $(\mathfrak{l}_{s;l}^{\mathfrak{so},2})$ be the maximal $\mathfrak{sl}_2$-submodule of $S^2(F_{s;l})$ $(\wedge^2(E_{s;l}))$. Then from (8.16)

$$(8.18) \qquad \mathfrak{l}^{\mathfrak{sp}}(V_{s;l}^{\mathfrak{sp}}) \cong \mathfrak{n}_{(s-l,s),(s-l,s)} \oplus \mathfrak{l}_{s;l}^{\mathfrak{sp},1} \oplus \mathfrak{l}_{s;l}^{\mathfrak{sp},2} \cong \mathbb{K} \oplus \mathfrak{l}_{s;l}^{\mathfrak{sp},1} \oplus \mathfrak{l}_{s;l}^{\mathfrak{sp},2}$$

$$(8.19) \qquad \mathfrak{l}^{\mathfrak{so}}(V_{s;l}^{\mathfrak{so}}) \cong \mathfrak{n}_{(s-l,s),(s-l,s)} \oplus \mathfrak{l}_{s;l}^{\mathfrak{so},1} \oplus \mathfrak{l}_{s;l}^{\mathfrak{so},2} \cong \mathbb{K} \oplus \mathfrak{l}_{s;l}^{\mathfrak{so},1} \oplus \mathfrak{l}_{s;l}^{\mathfrak{so},2}$$

Let us describe $\mathfrak{l}_{s;l}^{\mathfrak{sp},1}$, $\mathfrak{l}_{s;l}^{\mathfrak{sp},2}$ $\mathfrak{l}_{s;l}^{\mathfrak{so},1}$ and $\mathfrak{l}_{s;l}^{\mathfrak{so},2}$. Let $l = s - r$. In order to describe $\mathfrak{l}_{s;l}^{\mathfrak{sp},1}$ and $\mathfrak{l}_{s;l}^{\mathfrak{so},1}$ note that $\mathfrak{sl}_2$-submodules $S^2(E_{s;l})$ and $\wedge^2(E_{s;l})$) are decomposed into the irreducible $\mathfrak{sl}_2$-modules as

follows:

$$\text{(8.20)} \qquad\qquad S^2(E_{s;l}) = \bigoplus_{i=0}^{[\frac{l}{2}]} \Pi_{2l-4i};$$

$$\text{(8.21)} \qquad\qquad \wedge^2(E_{s;l}) = \bigoplus_{i=0}^{[\frac{l-1}{2}]} \Pi_{2l-2-4i}.$$

(see, for example, [20]).

The submodule Π_{2l} of the largest dimension in $S^2(E_{s;l})$ is generated by the elements of highest or lowest degree in $S^2(E_{s;l})$, which are equal to $2s$ and $2s - 2l$, respectively. The range of degrees for each next submodule in the decomposition (8.20) is shrunk by 2 from both left and right sides, i.e. the submodule Π_{2l-4i} has degrees varying from $2s - 2l + 2i$ and $2s - 2i$. The submodule $\mathfrak{l}_{s;l}^{\mathfrak{sp},1}$ is equal to the direct sum of the submodules from the decomposition (8.20) for which all degrees are non-negative, i.e. for which $\max\{0, l - [s]\} \leq i \leq \min\{[\frac{l}{2}], [s]\}$. Since $l \leq 2[s]$, we have that $[\frac{l}{2}] \leq [s]$. Therefore

$$\text{(8.22)} \qquad\qquad \mathfrak{l}_{s;l}^{\mathfrak{sp},1} = \bigoplus_{i=\max\{0, l-[s]\}}^{[\frac{l}{2}]} \Pi_{2l-4i},$$

Note that from the condition $l \leq 2[s]$ and the last formula it follows that $\mathfrak{l}_{s;l}^{\mathfrak{sp},1} \neq 0$.

In order to get $\mathfrak{l}_{l;s}^{\mathfrak{sp},2}$ we have to replace s by $l - s$ in the right-hand side of (8.22), i.e.

$$\mathfrak{l}_{l;s}^{\mathfrak{sp},2} = \bigoplus_{i=\max\{0, l-[l-s]\}}^{[\frac{l}{2}]} \Pi_{2l-4i},$$

Again from the fact that $l < 2[s]$ it follows that

$$\text{(8.23)} \qquad\qquad \mathfrak{l}_{rs}^{\varepsilon,2} = \begin{cases} \mathbb{K} & \text{if } l \text{ is even and } s = \frac{l}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Besides, by (8.8) one has $\mathfrak{n}_{(s-l,s),(s-l,s)} = \mathbb{K}$. Substituting this and relation (8.23) into (8.22) we get that in the case l is odd or $s \neq \frac{l}{2}$

$$\text{(8.24)} \qquad\qquad \mathfrak{l}(V_{s;l}^{\mathfrak{sp}}) \cong \mathbb{K} \oplus \mathfrak{l}_{s;l}^{\mathfrak{sp},1},$$

where $\mathfrak{l}_{s;l}^{\mathfrak{sp},1}$ is as in (8.22). Finally, by (8.22) we get $\mathfrak{l}_{p;2p}^{\mathfrak{sp},1} = \mathbb{K}$ for any nonnegative integer p. From this, formula (8.23), and identification (8.18) it is easy to see that for any nonnegative integer p

$$\text{(8.25)} \qquad\qquad \mathfrak{l}^{\mathfrak{sp}}(V_{p;2p}^{\mathfrak{sp}}) \cong \mathfrak{sl}_2.$$

Similarly in the orthogonal case the submodule Π_{2l-2} of the largest dimension in $\wedge^2(E_{s;l})$ is generated by the elements of highest or lowest degree in $\wedge^2(E_{s;l})$, which is equal to $2s - 1$ and $2s - 2l + 1$ respectively. The range of degrees for each next submodule in the decomposition (8.21) is shrunk by 2 from both left and right sides, i.e. the submodule $\Pi_{2l-2-4i}$ has degrees varying from $2s - 2l + 1 + 2i$ to $2s - 1 - 2i$. The submodule $\mathfrak{l}_{s;l}^{\mathfrak{so},1}$ is equal to the direct sum of submodules from the decomposition (8.21) for which all degrees are non-negative, i.e. for which

$\max\{0, l - [s + \frac{1}{2}]\} \le i \le \min\{[\frac{l-1}{2}], [s - \frac{1}{2}]\}$. Since $l \le 2(s + \{s\}) - 1$, we have that $[\frac{l-1}{2}] \le [s - \frac{1}{2}]$. Thus

$$(8.26) \qquad\qquad \mathfrak{l}_{s;l}^{\mathfrak{so},1} = \bigoplus_{i=\max\{0, l-[s+\frac{1}{2}]\}}^{[\frac{l-1}{2}]} \Pi_{2l-2-4i}.$$

Note that from the condition $l \le 2(s + \{s\}) - 1$ and the last formula it follows that $\mathfrak{l}_{s;l}^{\mathfrak{so},1} \ne 0$.

In order to get $\mathfrak{l}_{s;l}^{\mathfrak{so},2}$ we have to replace s by $l - s$ in the right-hand side of (8.26), i.e.

$$\mathfrak{l}_{s;l}^{\mathfrak{so},2} = \bigoplus_{i=\max\{0, l-[l-s+\frac{1}{2}]\}}^{[\frac{l-1}{2}]} \Pi_{2l-2-4i}.$$

Again from the fact that $l \le 2(s + \{s\}) - 1$ it follows that

$$(8.27) \qquad\qquad \mathfrak{l}_{s;l}^{\mathfrak{so},2} = \begin{cases} \mathbb{K} & l \text{ is odd and } s = \frac{l}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly to (8.24) in the case l is even or $s \ne \frac{l}{2}$ we get that

$$(8.28) \qquad\qquad \mathfrak{l}^{\mathfrak{so}}(V_{s;l}^{\mathfrak{so}}) \cong \mathbb{K} \oplus \mathfrak{l}_{s;l}^{\mathfrak{so},1},$$

where $\mathfrak{l}_{s;l}^{\mathfrak{so},1}$ is as in (8.26). Finally, by (8.26) we get $\mathfrak{l}_{p-\frac{1}{2};2p-1}^{\mathfrak{so},1} = \mathbb{K}$ for any $p \in \mathbb{N}$. From this, formula (8.27), and identification (8.19) it is easy to see that

$$(8.29) \qquad\qquad \mathfrak{l}^{\mathfrak{so}}(V_{p-\frac{1}{2};2p-1}^{\mathfrak{so}}) \cong \mathfrak{sl}_2, \quad p \in \mathbb{N}.$$

8.3.2. Description of $\mathfrak{l}^{\mathfrak{sp}}(\mathcal{L}_m^{\mathfrak{sp}})$ and $\mathfrak{l}^{\mathfrak{so}}(\mathcal{L}_m^{\mathfrak{so}})$.

Note that $\mathfrak{l}^{\mathfrak{sp}}(\mathcal{L}_m^{\mathfrak{sp}}) \subset \mathfrak{n}(\mathcal{L}_m^{\mathfrak{sp}}, \mathcal{L}_m^{\mathfrak{sp}}) \cong \mathbb{K}\mathrm{Id}$, but $\mathrm{Id} \notin \mathfrak{sp}(\mathcal{L}_m^{\mathfrak{sp}})$. In the same way $\mathfrak{l}^{\mathfrak{so}}(\mathcal{L}_m^{\mathfrak{so}}) \subset \mathfrak{n}(\mathcal{L}_m^{\mathfrak{so}}, \mathcal{L}_m^{\mathfrak{so}}) \cong \mathbb{K}\mathrm{Id}$, but $\mathrm{Id} \notin \mathfrak{so}(\mathcal{L}_m^{\mathfrak{so}})$ Therefore

$$(8.30) \qquad\qquad \mathfrak{l}^{\mathfrak{sp}}(\mathcal{L}_m^{\mathfrak{sp}}) = 0, \quad \mathfrak{l}^{\mathfrak{so}}(\mathcal{L}_m^{\mathfrak{so}}) = 0.$$

8.3.3. Description of $\mathfrak{n}(V_{s_1;l_1}^{\mathfrak{sp}}, V_{s_2;l_2}^{\mathfrak{sp}})$ (or of $\mathfrak{n}(V_{s_1;l_1}^{\mathfrak{so}}, V_{s_2;l_2}^{\mathfrak{so}})$).

By Remark 7.2 we are interested only in the cases when both s_1 and s_2 are integers or both s_1 and s_2 belong to $\frac{1}{2}\mathbb{Z}_{\mathrm{odd}}$. Set $r_1 = s_1 - l_1$ and $r_2 = s_2 - l_2$. In the sequel ε will mean either $\mathfrak{sp}$ or $\mathfrak{so}$. Note that

$$\mathrm{Hom}(V_{s_1;l_1}^{\varepsilon}, V_{s_2;l_2}^{\varepsilon}) = \mathrm{Hom}(E_{s_1;l_1}, E_{s_2;l_2}) \oplus \mathrm{Hom}(F_{s_1;l_1}, F_{s_2;l_2}) \oplus \mathrm{Hom}(E_{s_1;l_1}, F_{s_2;l_2}) \oplus \mathrm{Hom}(F_{s_1;l_1}, E_{s_2;l_2}).$$

Therefore,

$$(8.31) \quad \mathfrak{n}(V_{s_1;l_1}^{\varepsilon}, V_{s_2;l_2}^{\varepsilon}) \cong \mathfrak{n}_{(r_1,s_1),(r_2,s_2)} \oplus \mathfrak{n}_{(-s_1,-r_1),(-s_2,-r_2)} \oplus \mathfrak{n}_{(r_1,s_1),(-s_2,-r_2)} \oplus \mathfrak{n}_{(-s_1,-r_1),(r_2,s_2)}$$

Often many terms in the right-hand side of (8.31) vanish. First, let us analyze the term $\mathfrak{n}_{(r_1,s_1),(-s_2,-r_2)}$. By (8.9) if $\mathfrak{n}_{(r_1,s_1),(-s_2,-r_2)} \ne 0$, then $r_1 \le -s_2$ and $s_1 \le -r_2$ or, equivalently, $s_1 - l_1 \le -s_2$ and $s_1 \le l_2 - s_2$. In the symplectic case, in addition we have $l_1 \le 2[s_1]$ and $l_2 \le 2[s_2]$. Using all four

inequalities we get that $s_1 = s_2$, s_1, s_2 are integers and $l_1 = l_2 = 2s_1$. This together with (8.8) implies that

$$(8.32) \qquad \mathfrak{n}_{(r_1,s_1),(-s_2,-r_2)} = \mathfrak{n}_{(s_1-l_1,s_1),(-s_2,l_2-s_2)} = \begin{cases} \mathbb{K} & \text{if } s_1 = s_2, \ s_1, s_2 \in \mathbb{Z}, \ l_1 = l_2 = 2s_1 \\ 0 & \text{otherwise.} \end{cases}$$

In the orthogonal case, in addition to $s_1 - l_1 \le -s_2$ and $s_1 \le l_2 - s_2$ we have $l_1 \le 2(s_1 + \{s_1\}) - 1$ and $l_2 \le 2(s_2 + s_2) - 1$. Using all four inequalities we get that $s_1 = s_2$, s_1, s_2 belongs to $\frac{1}{2}\mathbb{Z}_{\text{odd}}$, and $l_1 = l_2 = 2s_1$. So, we will have the same formula as (8.32) except that s_1, s_2 belongs to $\frac{1}{2}\mathbb{Z}_{\text{odd}}$.

Further, by (8.9)

$$(8.33) \qquad \mathfrak{n}_{(-s_1,-r_1),(-s_2,-r_2)} = 0 \text{ if and only if } r_1 < r_2 \text{ or } s_1 < s_2.$$

Substituting formulas (8.32) (or its analog for the orthogonal case), (8.9), (8.33) into (8.31) and using (8.15) for nonzero terms, one gets an explicit expression for $\mathfrak{n}(V^{\mathfrak{sp}}_{s_1;l_1}, V^{\varepsilon}_{s_2;l_2})$. In particular, if $(r_1, s_1, r_2, s_2) \ne (-s, s, -s, s)$, then at least two of the first three terms in the right-hand side of (8.31) vanish. Finally, it is easy to see that

$$(8.34) \qquad \mathfrak{n}(V^{\varepsilon}_{s;2s}, V^{\varepsilon}_{s;2s}) \cong \mathfrak{gl}_2, \quad \varepsilon = \mathfrak{sp} \text{ or } \mathfrak{so}.$$

8.3.4. *Description of $\mathfrak{n}(V^{\mathfrak{sp}}_{s;l}, \mathcal{L}^{\mathfrak{sp}}_m)$ and of $\mathfrak{n}(V^{\mathfrak{so}}_{s;l}, \mathcal{L}^{\mathfrak{so}}_m)$.*

By analogy with the previous case

$$(8.35) \qquad \mathfrak{n}(V^{\varepsilon}_{s;l}, \mathcal{L}^{\varepsilon}_m) = \mathfrak{n}_{(s-l,s),(-m,m)} + \mathfrak{n}_{(-s,l-s),(-m,m)}.$$

First, let us analyze the term $\mathfrak{n}_{(s-l,s),(-m,m)}$. By (8.9) if $\mathfrak{n}_{(s-l,s),(-m,m)} \ne 0$, then $s - l \le -m$ and $s \le m$. By Remark 7.2, $s \in \frac{1}{2}\mathbb{Z}_{\text{odd}}$ in the symplectic case and $s \in \mathbb{Z}$ in the orthogonal case, and therefore $l \le 2s - 1$ in both cases, which together with the previous inequalities implies that $-s + 1 \le s - l \le -m \le -s$, which is impossible. Consequently, for any admissible triple (l, s, m) we get $\mathfrak{n}_{(s-l,s),(-m,m)} = 0$, i.e.

$$(8.36) \qquad \mathfrak{n}(V^{\varepsilon}_{s;l}, \mathcal{L}_m) = \mathfrak{n}_{(-s,l-s),(-m,m)}, \quad \varepsilon = \mathfrak{so} \text{ or } \mathfrak{sp}.$$

Besides, $\mathfrak{n}_{(-s,l-s),(-m,m)} \ne 0$ if and only if $l - s \le m \le s$ and it can be computed using formula (8.15).

8.3.5. *Description of $\mathfrak{n}(\mathcal{L}^{\mathfrak{sp}}_{m_1}, \mathcal{L}^{\mathfrak{sp}}_{m_2})$ (or of $\mathfrak{n}(\mathcal{L}^{\mathfrak{so}}_{m_1}, \mathcal{L}^{\mathfrak{so}}_{m_2})$).*

By (8.9), both for $\varepsilon = \mathfrak{sp}$ and $\varepsilon = \mathfrak{so}$ we have

$$(8.37) \qquad \mathfrak{n}(\mathcal{L}^{\varepsilon}_{m_1}, \mathcal{L}^{\varepsilon}_{m_2}) = \mathfrak{n}_{(-m_1,m_1),(-m_2,m_2)} = \begin{cases} \mathbb{K} & \text{if } m_1 = m_2 \\ 0 & \text{if } m_1 \ne m_2. \end{cases}$$

8.3.6. *The case of tensor products.*

In order to use formula (8.13) more effectively it is worth also to say more about the space $\mathfrak{l}^{\mathfrak{sp}}(V \otimes \mathbb{K}^N)$, where V is a graded symplectic space which is also a nice symplectic $\mathfrak{sl}_2$-module and $V \otimes \mathbb{K}^N$ inherits from V the structure of a graded symplectic space and of a nice symplectic $\mathfrak{sl}_2$-module in a natural way. From (8.12) it is easy to get the following natural identification:

$$(8.38) \qquad \mathfrak{sp}(V \otimes \mathbb{K}^N) \cong (\mathfrak{sp}(V) \otimes \mathbb{K}^N) \oplus (\mathfrak{gl}(V) \otimes \wedge^2 \mathbb{K}^N),$$

where $\wedge^2 K^N$ denotes a skew-symmetric square of $\mathbb{K}^N$. Consequently,

$$(8.39) \qquad \mathfrak{l}^{\mathfrak{sp}}(V \otimes \mathbb{K}^N) \cong (\mathfrak{l}^{\mathfrak{sp}}(V) \otimes \mathbb{K}^N) \oplus (\mathfrak{n}(V) \otimes \wedge^2 K^N).$$

Combining (8.10),(8.39), and the calculations of the subsections 8.3.1-8.3.5, one can get more compact explicit formula for the algebraic prolongation of a given symplectic symbol.

In particular, from (8.30) and (8.37) it follows that

$$(8.40) \qquad \mathfrak{l}^{\mathfrak{sp}}(\mathcal{L}_m^{\mathfrak{sp}} \otimes \mathbb{K}^N) \cong \wedge^2 K^N \cong \mathfrak{so}(N).$$

Now assume that $K = \mathbb{R}$. Let V be a nice symplectic $\mathfrak{sl}_2$-module such that as a graded space it coincides with $\mathcal{L}_m^{\mathfrak{sp}} \otimes \mathbb{R}^{N_+ + N_-}$ for some nonnegative integers N_+ and N_- and the degree -1 component of the corresponding embedding of $\mathfrak{sl}_2$ into $\mathfrak{sp}(V)$ is generated by the direct sum of $\tau_m^{\mathfrak{sp}}$ taken N_+ times and $-\tau_m^{\mathfrak{sp}}$ taken N_- times. Then similarly to above it can be shown that

$$(8.41) \qquad \mathfrak{l}(V) \cong \mathfrak{so}(N_+, N_-).$$

Further, fix two functions $N_+, N_- : \frac{1}{2}\mathbb{Z}_{\mathrm{odd}} \mapsto \mathbb{N} \cup \{0\}$ with finite support and assume that the symplectic symbol $\mathfrak{m}$ is generated by the direct sum of endomorphisms of type $\tau_m^{\mathfrak{sp}}$ and $-\tau_m^{\mathfrak{sp}}$, where $\tau_m^{\mathfrak{sp}}$ appears $N_+(m)$ times and $-\tau_m^{\mathfrak{sp}}$ appears $N_-(m)$ times in this sum for each $m \in \frac{1}{2}\mathbb{Z}_{\mathrm{odd}}$. These symbols correspond to curves in a Lagrangian Grassmannian satisfying condition (G) in the terminology of the previous papers of the second author with C. Li ([41, 42]). Then from (8.41) and (8.37) it follows that the non-effectiveness ideal of $\mathfrak{u}(\mathfrak{m})$ is equal to $\bigoplus_{m \in \mathbb{N}} \mathfrak{so}(N_+(m), N_-(m))$ and it is concentrated in the zero component of $\mathfrak{u}(\mathfrak{m})$. From this and Remark 8.1 it follows that in the parameterized case $\mathfrak{u}_+(\mathfrak{m}) = \bigoplus_{m \in \mathbb{N}} \mathfrak{so}(N_+(m), N_-(m))$ and, by Theorem 4.1 and Remark 4.3, if the normalization condition is fixed, then for any parametrized curve of flags with symbol $\mathfrak{m}$ there exists a unique principal bundle of moving frames with the structure group $\prod_{m \in \mathbb{N}} O(N_+(m), N_-(m))$ and the unique principal Ehresmann connection such that this connection satisfies the chosen normalization condition. This result was proved in [41, 42] (Theorem 1 in [41] and Theorems 1 and 3 in [42]) for specifically chosen normalization conditions. So, our main Theorem 4.1 gives more conceptual point of view on the constructions of those papers and clarifies them algebraically.

Finally, formulas (8.39)-(8.41) hold true if V is a graded orthogonal space and $\mathfrak{sp}$ is replaced by $\mathfrak{so}$. Also, the same conclusion as in the previous paragraph can be done if an orthogonal symbol is generated by the direct sum of endomorphisms of type $\tau_m^{\mathfrak{so}}$ and $-\tau_m^{\mathfrak{so}}$.

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