Endow M with the order topology. Then M is a 1-manifold which is Hausdorff but not paracompact, called the *long line*. M has a C^ω differential structure but no Riemannian metric. (See Koch and Puppe [1], Kneser and Kneser [1].)

10. Let L be the quotient space obtained from $(\mathbb{R} \times 1) \cup (\mathbb{R} \times 0)$ by identifying $(x,1)$ with $(x,0)$ if $x \neq 0$. Then L is a nonHausdorff 1-manifold, called the *line with two origins*. It has a C^ω differential structure.

***11.** Let $U \subset \mathbb{R}^2$ be a nonempty open set. Suppose given a C^r $(r > 0)$ vector field on U without zeros, such that each integral curve is closed in U. Let M be the identification space obtained by collapsing each integral curve to a point. Then M is a C^r 1-manifold, which can be non-Hausdorff. [Hint: Use small intervals transverse to the integral curves to construct charts.]

***12.** A manifold is metrizable, and has a complete metric, if and only if it is paracompact and Hausdorff. A connected metrizable manifold has a countable base. But there is a connected separable Hausdorff 2-manifold which is not paracompact, (the double of M in Exercise 7, Section 4.6).

***13.** A paracompact manifold is an absolute neighborhood retract (see Hanner [1]).

2. Differentiable Maps and the Tangent Bundle

From now on we shall frequently suppress notation for the differential structure on a manifold M.

Let M and N be C^r manifolds and $f: M \to N$ a map. A pair of charts (φ,U) for M and (ψ,V) for N is *adapted to* f if $f(U) \subset V$. In this case the map

$$\psi f \varphi^{-1}: \varphi(U) \to \psi(V)$$

is defined; we call it the *local representation* of f in the given charts, *at the point* x if $x \in U$.

The map f is called *differentiable* at x if it has a local representation at x which is differentiable. This definition makes sense since a local representation is a map between open sets in Cartesian spaces. Similarly, f is *differentiable of class* C^r if it has C^r local representations at all points.

If f is C^r then every local representation is C^r. To see this, let (φ,U) and (ψ,V) be a pair of charts adapted to f, and suppose f is C^r. To prove $\psi f \varphi^{-1} C^r$, let $y \in \varphi(V)$ be any point; put $x = \varphi^{-1}(y)$. Let (φ_0,U_0) and (ψ_0,V_0) be an adapted pair of charts giving f the C^r local representation $\psi_0 f \varphi_0^{-1}$ at x. By replacing U_0 and V_0 by smaller open sets, if necessary, we can arrange that $U_0 \subset U$ and $V_0 \subset V$. Then

$$\psi f \varphi^{-1} = (\psi \psi_0^{-1})(\psi_0 f \varphi_0^{-1})(\varphi_0 \varphi^{-1})$$

in $\varphi(U_0)$. The first and third maps on the right are C^r since they are coordinate changes. Hence $\psi f \varphi^{-1} | \varphi(U_0)$ is the composition of C^r maps and so is C^r. This proves that $\psi f \varphi^{-1}$ is C^r in some neighborhood of every point, and so it is C^r.

Let $f: M \to N$ and $g: N \to P$ be C^r maps between C^r manifolds. It is easy to verify, using local representations, that the composition $gf: M \to P$ is also C^r. The identity map and all constant maps are C^r. There is evidently a category of C^r manifolds and C^r maps.

An isomorphism in the C^r category is called a C^r *diffeomorphism*. (If $r = 0$ this means a homeomorphism.) Explicitly, a C^r diffeomorphism $f: M \to N$ is a C^r map between C^r manifolds M and N which is a homeomorphism, and whose inverse $f^{-1}: N \to M$ is also of class C^r. If such a map exists we call M and N C^r *diffeomorphic* manifolds and write $M \approx N$. This is the basic equivalence relation of differential topology.

Lest the reader lose heart at the prospect of an infinite sequence of equivalence relations, one for each r, we hasten to point out that there is no essential difference between C^r and C^s for $1 \leqslant r < s \leqslant \infty$ (or even $s = \omega$, but that is much more difficult). In Chapter 3 we shall see that every C^r manifold is C^r diffeomorphic to a C^ω manifold, and the latter is unique up to C^ω diffeomorphism; and any C^r map can be approximated by C^ω maps.

There is, however, an unbridgeable gap between C^0 and C^1. In fact one of the most fascinating topics in differential topology began with the discoveries by Kervaire [1] and Smale [1] of compact manifolds having no differential structure whatever. (It is known that such a "nonsmoothable" manifold must have dimension at least 4; explicit examples are known in dimension 8.)

A basic task of differential topology is to find methods for deciding whether two given manifolds diffeomorphic. Of course diffeomorphic manifolds are homeomorphic, and have the same homotopy type. Therefore the diffeomorphism problem usually takes the form: what more do we need to know about two manifolds, in addition to their having the same homotopy type, to guarantee that they are diffeomorphic?

Often a differential invariant turns out to be a topological or homotopy type invariant. (The classic example is the sum of the indices of zeros of a vector field on a compact smooth manifold, which turns out to equal the Euler characteristic.) Such an invariant cannot distinguish between non-diffeomorphic manifolds which are homeomorphic. On the other hand, when a differential invariant is a homotopy invariant as well, it is easier to compute.

One of the most important differential invariants is the tangent bundle. In later chapters we will study the tangent bundle in some detail; here we merely give its definition (as a manifold) and the definition of the tangent of a map.

Let (M,Φ) be a C^{r+1} manifold, $0 \leqslant r \leqslant \omega$, where $\infty + 1 = \infty$ and $\omega + 1 = \omega$, with $\Phi = \{\varphi_i, U_i\}_{i \in A}$. Intuitively speaking, a "tangent vector" to M at $x \in M$ is simply a vector in $\mathbb{R}^n$ together with a chart which identifies each point near x with a point of $\mathbb{R}^n$.

A tangent vector should be an object independent of any particular chart, however, so we make the following definition. A *tangent vector* to M is an

equivalence class $[x,i,a]$ of triples

$$(x,i,a) \in M \times \Lambda \times \mathbb{R}^n$$

under the equivalence relation:

$$[x,i,a] = [y,j,b]$$

if and only if $x = y$ and

$$D(\varphi_j\varphi_i^{-1})(\varphi_i(x))a = b.$$

In other words, the derivative of the coordinate change at $\varphi_i(x)$ sends a to b. That this is an equivalence relation follows from the rules for derivatives of compositions and inverses.

The set of all tangent vectors is TM, the *tangent bundle* of M. The map

$$p = p_M : TM \to M,$$
$$[x,i,a] \mapsto x$$

is well defined. For any subset $A \subset M$ we put $p^{-1}(A) = T_A M$; also $p^{-1}(x) = M_x$ for $x \in M$. If $U \subset M$ is open then $(U,\Phi|U)$ is also a C^{r+1} manifold, and we make the harmless identification $T_U M = TU$.

For any chart $(\varphi_i,U_i) \in \Phi$ there is a well defined bijective map

$$T\varphi_i : TU_i \to \varphi_i(U_i) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n,$$
$$[x,i,a] \mapsto (\varphi_i(x),a).$$

The map

$$(T\varphi_j)(T\varphi_i)^{-1} : \varphi_i(U_i \cap U_j) \times \mathbb{R}^n \to \varphi_j(U_i \times U_j) \times \mathbb{R}^n$$

is the homeomorphism

$$(y,a) \mapsto (\varphi_j\varphi_i^{-1}(y), D(\varphi_j\varphi_i^{-1})(y)a).$$

It follows that TM has a topology making each $T\varphi_i$ a homeomorphism, and this topology is unique. Moreover, since $(T\varphi_j)(T\varphi_i)^{-1}$ is a C^r diffeomorphism, the set of charts $\{T\varphi_i, TU_i\}_{i \in \Lambda}$ is a C^r atlas on TM. In this way TM is a C^r manifold. The projection map $p: TM \to M$ is C^r. The charts $(T\varphi_i, TU_i)$ are called *natural charts* on TM.

Let $x \in U_i$. The map $T\varphi_{ix} : M_x \to \mathbb{R}^n$, defined as the composition

$$M_x \subset TU_i \xrightarrow{T\varphi} \varphi_i(U_i) \times \mathbb{R}^n \to \mathbb{R}^n,$$

is a bijection; hence it induces an n-dimensional vector space structure on M_x. *This structure is independent of* i, since if $x \in U_j$,

$$(T\varphi_{jx})(T\varphi_{ix})^{-1} = D(\varphi_j\varphi_i^{-1})(\varphi_i x)$$

which is a linear automorphism of $\mathbb{R}^n$. In this way M_x becomes a vector space, the *tangent space to* M *at* x. Thus TM is the disjoint union of the vector spaces M_x. It is a bundle of vector spaces, or "vector bundle." This aspect of TM will be emphasized in later chapters.

The simplest kind of tangent bundle is that of an open set $W \subset \mathbb{R}^q$. In this case we identify TW with $W \times \mathbb{R}^q$ via the inclusion chart $\varphi: W \to \mathbb{R}^q$ and the corresponding natural chart on TW. The projection $TW \to W$ is just the natural projection $W \times \mathbb{R}^q \to W$. If M is a submanifold of $\mathbb{R}^3$ we can think of tangent vectors to M as arrows and M_x as a plane, as in Figure 1–2.

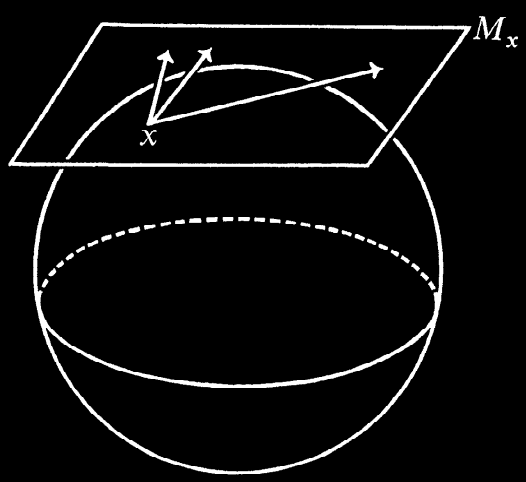


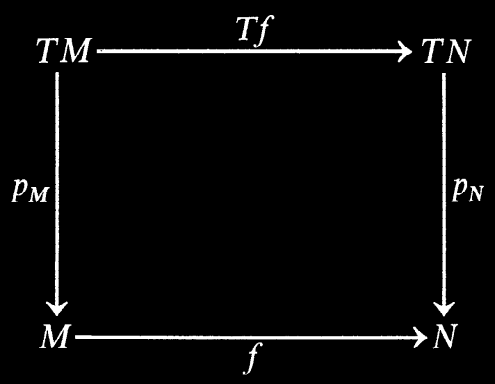
Figure 1–2. Tangent vectors to $M = S^2 \subset \mathbb{R}^3$.

Let $f: M \to N$ be a C^{r+1} map, $0 \leqslant r \leqslant \omega$. A C^r map $Tf: TM \to TN$ is defined as follows: a local representation of Tf in natural charts on TM and TN is the derivative of the corresponding local representation of f. More explicitly, let $\varphi_i: U_i \to \mathbb{R}^m$, $\psi_j: V_j \to \mathbb{R}^n$ be charts for M, N with $f(U_i) \subset V_j$. An application of the chain rule shows that the C^r map

$$(Tf)_{ij}: TU_i \to TV_j,$$
$$[x,i,a] \mapsto [f(x),j,D(\psi_j f \varphi_i^{-1})(\varphi_i x)a]$$

is independent of i, j. Thus there is a well defined map $Tf: TM \to TN$ which coincides with $(Tf)_{ij}$ on TU_i.

If $f(x) = y$ then Tf maps M_x into N_y, and the restriction of Tf is a *linear* map: $T_x f: M_x \to N_y$.

In the natural charts this is just the derivative at x of the corresponding local representation of f. Thus $T_x f$ may be thought of as the derivative of f at x. Note, however, that its domain and range depend on x.

Using natural charts one sees that the diagram