MEASURE CONTRACTION PROPERTIES OF CONTACT SUB-RIEMANNIAN MANIFOLDS WITH SYMMETRY

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Abstract. Measure contraction properties are generalizations of the notion of Ricci curvature lower bounds in Riemannian geometry to more general metric measure spaces. In this paper, we give sufficient conditions for a contact sub-Riemannian manifold with a one-parameter family of symmetries to satisfy this property. Moreover, in the special case where the quotient of the contact sub-Riemannian manifold by the symmetries is Kähler, the sufficient conditions are defined by a combination of the holomorphic sectional curvature and the Ricci curvature. This generalizes the earlier work in [1] for the three dimensional case and in [10] for the Heisenberg group. Finally, we also prove a version of Bonnet-Myer's Theorem in our setting.

1. Introduction

In recent years, there are lots of efforts in generalizing the notion of Ricci curvature lower bounds in Riemannian geometry and its consequences to more general metric measure spaces. One of them is the work of [16, 17, 22, 23] where the notion of curvature-dimension conditions was introduced. These conditions are generalizations of Ricci curvature lower bounds to length spaces equipped with a measure (length spaces are metric spaces on which the notion of geodesics is defined). In [20], it was shown that the curvature-dimension conditions coincide with the pre-existing notion of Ricci curvature lower bounds in the case of Finsler manifolds.

On the contrary, it was shown in [10] that the curvature-dimension conditions defined using the theory of optimal transportation are not satisfied on the Heisenberg group, the simplest sub-Riemannian manifold. (Note however that a type of curvature-dimension conditions were defined in [4, 5] using a sub-Riemannian version of the Bochner formula). It was also shown in [10] that the Heisenberg group satisfies

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another generalization of Ricci curvature lower bounds to length spaces called measure contraction properties [22, 23, 19].

Measure contraction properties $\mathcal{MCP}(k,n)$ are essentially defined by the rate of contraction of volume along geodesics inspired by the classical Bishop volume comparison theorem. In the Riemannian case, $\mathcal{MCP}(k,n)$ is equivalent to the conditions that the Ricci curvature is bounded below by k and the dimension is bounded above by n. In [10], it was shown that the Heisenberg group of dimension $2n+1$ satisfies the condition $\mathcal{MCP}(0,2n+3)$.

In [1], the case of a general three dimensional contact sub-Riemannian manifold was studied. In particular, it was shown in [1] that a three-dimensional Sasakian manifold satisfies $\mathcal{MCP}(0,5)$ if and only if the Tanaka-Webster curvature is bounded below by 0. Moreover, conditions called generalized measure contraction properties $\mathcal{MCP}(k;2,3)$ were also defined in [1]. It was shown that a three-dimensional Sasakian manifold satisfies this condition if and only if the Tanaka-Webster curvature is bounded below by k. Note also that there is a sub-Riemannian version of Bishop theorem which was proved in [7, 2]. Unlike the Riemannian case, this is very different from the generalized measure contraction properties $\mathcal{MCP}(k;2,3)$ (see [2]). This is essentially due to the fact that any neighborhood contains points which are joined by more than one minimizing sub-Riemannian geodesics.

In this paper, we generalize the results in [10, 1] to any contact sub-Riemannian manifolds with symmetry. We introduce new generalized measure contraction properties $\mathcal{MCP}(k_1,k_2;N-1,N)$ and discuss when a contact sub-Riemannian manifold with symmetry satisfies them.

Next, we state the condition $\mathcal{MCP}(k_1,k_2;N-1,N)$ and some simple consequences of the main results. Let M be a sub-Riemannian manifold (see Section 2 for a discussion on some basic notions in sub-Riemannian geometry). For simplicity, we assume that M satisfies the following property: given any point x_0 in M, there is a set of Lebesgue measure zero such that any point outside the set is connected to x_0 by a unique length minimizing sub-Riemannian geodesic. By the result in [6], this is satisfied by all contact sub-Riemannian manifolds.

Let $t\mapsto\varphi_t(x)$ be the unique geodesic starting from x and ending at x_0. This defines a 1-parameter family of Borel maps. Let d be the subriemannian distance and let μ be a Borel measure. The following is the original measure contraction property studied in [22, 23, 19]:

A metric measure space (M,d,μ) satisfies $\mathcal{MCP}(0,N)$ if

$$\mu(\varphi_t(U))\geq(1-t)^N\mu(U)$$

for each point x_0 and each Borel set U.

Note that the condition $\mathcal{MCP}(0, N)$ implies the volume doubling property of μ and a local Poincaré inequality (see [22, 23, 19]). By combining this with the work of [8], this proves the Harnack inequality and hence the Liouville property for the subriemannian analogue of harmonic functions.

Next, we introduce the new generalized measure contraction properties $\mathcal{MCP}(k_1, k_2; N-1, N)$: A metric measure space (M, d, μ) satisfies $\mathcal{MCP}(k_1, k_2; N-1, N)$ if, for each point x_0 and each Borel set U,

$$\mu(\varphi_t(U)) \geq \int_U \frac{(1-t)^{N+2}\mathcal{M}_1(k_2 d^2(x, x_0), t)\mathcal{M}_2^{N-3}(k_1 d^2(x, x_0), t)}{\mathcal{M}_1(k_2 d^2(x, x_0), 0)\mathcal{M}_2^{N-3}(k_1 d^2(x, x_0), 0)} d\mu(x),$$

where

$$\mathcal{D}(k, t) = \sqrt{|k|}(1-t),$$

$$\mathcal{M}_1(k, t) = \begin{cases} \frac{2-2\cos(\mathcal{D}(k,t)) - \mathcal{D}(k,t)\sin(\mathcal{D}(k,t))}{\mathcal{D}(k,t)^4} & \text{if } k > 0 \\ \frac{1}{12} & \text{if } k = 0 \\ \frac{2-2\cosh(\mathcal{D}(k,t)) + \mathcal{D}(k,t)\sinh(\mathcal{D}(k,t))}{\mathcal{D}(k,t)^4} & \text{if } k < 0, \end{cases}$$

$$\mathcal{M}_2(k, t) = \begin{cases} \frac{\sin(\mathcal{D}(k,t))}{\mathcal{D}(k,t)} & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ \frac{\sinh(\mathcal{D}(k,t))}{\mathcal{D}(k,t)} & \text{if } k < 0. \end{cases}$$

Note, in particular, that $\mathcal{MCP}(0, 0; N-1, N)$ is the same as $\mathcal{MCP}(0, N+2)$. If $k_1 \geq 0$ and $k_2 \geq 0$, then $\mathcal{MCP}(k_1, k_2; N-1, N)$ implies $\mathcal{MCP}(0, N+2)$. The reason for the notations in the conditions $\mathcal{MCP}(k_1, k_2; N-1, N)$ is clarified by Theorem 1.1 below.

Next, we state a simple consequence of the main results. For this, we let M be a contact sub-Riemannian manifold with symmetry and of dimension $2n+1$. This means that the sub-Riemannian structure on M is defined by a contact distribution $\mathcal{D}$ and there is a sub-Riemannian isometry V_0 which is transversal to $\mathcal{D}$. We can extend the sub-Riemannian metric on M to a Riemannian one by the conditions that the vector field V_0 is of unit length and is orthogonal to $\mathcal{D}$. The corresponding Riemannian measure is denoted by μ.

Let $\langle \cdot, \cdot \rangle$ be the sub-Riemannian metric and let α be the contact form of $\mathcal{D}$ which satisfies the condition $\alpha(V_0) = 1$. Let J be the operator on $\mathcal{D}$ defined by $d\alpha(v, w) = \langle Jv, w \rangle$, where v and w are vectors in $\mathcal{D}$. Assume that the quotient $\widetilde{M}$ of M by the flow of V_0 is a manifold. Then $\langle \cdot, \cdot \rangle$ and J descend to a Riemannian metric and a operator on $\widetilde{M}$, still denoted by $\langle\cdot,\cdot\rangle$ and J, respectively. Finally, assume that $(\langle\cdot,\cdot\rangle, J)$ defines a Kähler structure on $\widetilde{M}$.

Theorem 1.1. *Assume that the Riemann curvature tensor* $\mathbf{Rm}$ *of the Kähler manifold* $\widetilde{M}$ *satisfies*

$$\langle \mathbf{Rm}(Jv,v)v, Jv\rangle \geq k_1|v|^4$$

and

$$\mathbf{Rc}(v,v) - \frac{1}{|v|^2}\langle \mathbf{Rm}(Jv,v)v, Jv\rangle \geq (2n-2)k_2|v|^2.$$

for all tangent vectors v. Then the metric measure space (M,d,μ) satisfies $\mathcal{MCP}(k_1,k_2;2n,2n+1)$, where d is the sub-Riemannian distance of M.

Remark that the first condition in Theorem 1.1 says that the holomorphic sectional curvature of $\widetilde{M}$ is bounded below by k_1. As a corollary, we have the following result of [10] mentioned above as a special case.

Theorem 1.2. [10] *The Heisenberg group of dimension n equipped with the standard sub-Riemannian distance d and the Lebesgue measure μ satisfies $\mathcal{MCP}(0,0;2n,2n+1) = \mathcal{MCP}(0,2n+3)$.*

The complex Hopf fibration $U(1) \to M = S^{2n+1} \to \widetilde{M} = \mathbb{CP}^n$ can be equipped with a sub-Riemannian metric such that M becomes a contact sub-Riemannian manifold with symmetry. In this case, V_0 is the infinitesimal generator of $U(1)$-action and the induced Riemannian metric on $\widetilde{M}$ is the Fubini-Study metric (see [18]).

Theorem 1.3. *The complex Hopf fibration equipped with the above sub-Riemannian distance d and the measure μ satisfies the condition $\mathcal{MCP}(4,1;2n,2n+1)$. In particular, it satisfies $\mathcal{MCP}(0,2n+3)$.*

We also remark that the estimates for the proof of Theorem 1.2 and 1.3 are sharp (see Corollary 2.1, 2.2, and 2.3 for more detail).

Finally, we also prove a Bonnet-Myer's type theorem in our setting.

Theorem 1.4. *Assume that the Riemann curvature tensor* $\mathbf{Rm}$ *of the Kähler manifold* $\widetilde{M}$ *satisfies*

$$\langle \mathbf{Rm}(Jv,v)v, Jv\rangle \geq k_1|v|^4$$

for some positive constant k_1. Then the diameter of the manifold M with respect to the corresponding sub-Riemannian metric is less than or equal to $\frac{2\pi}{\sqrt{k_1}}$.

On the other hand, if

$$\mathbf{Rc}(v,v) - \frac{1}{|v|^2}\langle \mathbf{Rm}(Jv,v)v, Jv\rangle \geq (2n-2)k_2|v|^2,$$

for some positive constant k_2, then the diameter of the manifold M with respect to the corresponding sub-Riemannian metric is less than or equal to $\frac{\pi}{\sqrt{k_2}}$.

In the next section, some basic notions in sub-Riemannian geometry will be recalled and the main results of the paper will be stated. The rest of the sections will be devoted to the proof of the main results.

2. The Main Results

In this section, we recall various notions in sub-Riemannian geometry which are needed and state the main results of this paper. A sub-Riemannian manifold is a triple $(M, \mathcal{D}, \langle\cdot,\cdot\rangle)$, where M is a manifold of dimension n, $\mathcal{D}$ is a sub-bundle of the tangent bundle TM, and $\langle\cdot,\cdot\rangle$ is a smoothly varying inner product defined on $\mathcal{D}$. The sub-bundle $\mathcal{D}$ and the inner product $\langle\cdot,\cdot\rangle$ are commonly known as a distribution and a sub-Riemannian metric, respectively. A curve $\gamma(\cdot)$ is horizontal if $\dot\gamma(t)$ is contained in $\mathcal{D}$ for all time t. The length $l(\gamma)$ of a horizontal curve γ can be defined as in the Riemannian case:

$$l(\gamma) = \int_0^1 |\dot\gamma(t)|dt.$$

Assume that the distribution $\mathcal{D}$ satisfies the following bracket generating or Hörmander condition: the sections of $\mathcal{D}$ and their iterated Lie brackets span each tangent space. Under this assumption and that the manifold M is connected, Chow-Rashevskii Theorem (see [18]) guarantees that any two given points on the manifold M can be connected by a horizontal curve. Therefore, we can define the sub-Riemannian distance d as

$$d(x_0, x_1) = \inf_{\gamma\in\Gamma} l(\gamma), \tag{2.1}$$

where the infimum is taken over the set Γ of all horizontal paths $\gamma : [0,1] \to M$ which connect x_0 with x_1: $\gamma(0) = x_0$ and $\gamma(1) = x_1$. The minimizers of (2.1) are called length minimizing geodesics (or simply geodesics). As in the Riemannian case, reparametrizations of a geodesic is also a geodesic. Therefore, we assume that all geodesics have constant speed.

In this paper, we will focus on contact sub-Riemannian manifolds meaning that the distribution $\mathcal{D}$ is given by the kernel of a 1-form α, called a contact form, defined by the condition that the restriction of $d\alpha$ to $\mathcal{D}$ is non-degenerate. We also assume that there is sub-Riemannian isometry V_0 on M which is transversal to $\mathcal{D}$. The sub-Riemannian isometry V_0 is a vector field for which the flow preserves the distribution $\mathcal{D}$ and the length of horizontal vectors. For simplicity, we will also assume that the quotient of M by the flow of V_0 is a manifold denoted by $\widetilde{M}$. Let μ be the volume form on M defined by the condition $\mu(V_0, v_1, ..., v_{n-1}) = 1$, where $v_1, ..., v_{n-1}$ is any orthonormal family of horizontal vectors. The measure corresponding to μ will also be denoted by the same symbol.

Since the distribution is contact, the sub-Riemannian distance $x \mapsto d(x, x_0)$ is locally semi-concave on $M - \{x_0\}$ by the result of [6]. In particular, the sub-Riemannian distance is differentiable almost everywhere. Therefore, outside a set of measure zero, the points can be connected to x_0 by a unique length minimizing sub-Riemannian geodesic. This also defines a family of Borel maps $\varphi_t : M \to M$ such that $t \mapsto \varphi_t(x)$ is the unique length minimizing geodesic connecting x and x_0.

Let $\langle \cdot, \cdot \rangle$ and $|\cdot|$ be the sub-Riemannian metric and the corresponding norm. Recall that the restriction of $d\alpha$ to the distribution $\mathcal{D}$ is non-degenerate. This defines an invertible endomorphism $J : \mathcal{D} \to \mathcal{D}$ by $d\alpha(v, w) = \langle Jv, w \rangle$, where v and w are contained in $\mathcal{D}$. Both the sub-Riemannian metric and the operator J descend to the manifold $\widetilde{M}$. The corresponding Riemannian metric and operator are denoted by the same symbols.

Theorem 2.1. *Assume that the tensor J is parallel and the Riemann curvature tensor $\mathbf{Rm}$ satisfies*

$$\langle \mathbf{Rm}(Jv, v)v, Jv \rangle \geq k_1 |v|^2 |Jv|^2$$

and

$$\mathbf{Rc}(v, v) - \frac{1}{|Jv|^2} \langle \mathbf{Rm}(Jv, v)v, Jv \rangle \geq (2n - 2) k_2 |v|^2.$$

Then, for any Borel set U,

$$\mu(\varphi_t(U)) \geq \frac{(1-t)^{2n+3}}{\lambda_1 \lambda_2} \int_U \frac{\mathcal{M}_1(\mathbf{k_1}(x), t) \mathcal{M}_2^{2n-2}(\mathbf{k_2}(x), t)}{\mathcal{M}_1(\mathbf{k_1}(x), 0) \mathcal{M}_2^{2n-2}(\mathbf{k_2}(x), 0)} d\mu(x),$$

where

$$
\begin{aligned}
&f(x) = d(x,x_0),\\
&\mathbf{k_1}(x) = f(x)^2\left(k_1 + \frac{(V_0f)^2(x)}{\lambda_2^2}\right),\\
&\mathbf{k_2}(x) = f(x)^2\left(k_2 - \frac{(V_0f)^2(x)}{4(2n-2)}\left(\lambda_3 + 2\lambda_1^2\right)\right),
\end{aligned}
$$

λ_1 *and* λ_2 *are upper bounds of the operator norms of* J *and* J^{-1}, *respectively,*

λ_3 *is an upper bound of* $\boldsymbol{tr}(J^2)$.

Therefore, under the above assumption, the metric measure space (M,d,μ) *satisfies* $\mathcal{MCP}(k_1,k_2;2n,2n+1)$. *In particular, if* k_1 *and* k_2 *are non-negative, then it satisfies* $\mathcal{MCP}(0,2n+3)$.

If, in addition to the assumptions of Theorem 2.1, we assume that the operator J satisfies $J^2=-I$ (i.e. the manifold $\widetilde{M}$ is Kähler), then Theorem 2.1 specializes to

Corollary 2.1. *Assume that* $(\widetilde{M}, \langle\cdot,\cdot\rangle, J)$ *defines a Kähler manifold and that the Riemann curvature tensor* $\mathbf{Rm}$ *satisfies*

$$\langle \mathbf{Rm}(Jv,v)v,Jv\rangle \geq k_1|v|^4$$

and

$$\mathbf{Rc}(v,v) - \frac{1}{|v|^2}\langle \mathbf{Rm}(Jv,v)v,v\rangle \geq (2n-2)k_2|v|^2.$$

Then

$$\mu(\varphi_t(U)) \geq \int_U \frac{(1-t)^{2n+3}\mathcal{M}_1(\mathbf{k_1}(x),t)\mathcal{M}_2^{2n-2}(\mathbf{k_2}(x),t)}{\mathcal{M}_1(\mathbf{k_1}(x),0)\mathcal{M}_2^{2n-2}(\mathbf{k_2}(x),0)}d\mu(x)$$

for any Borel set U, *where*

$$\mathbf{k_1}(x) = f(x)^2(k_1 + (V_0f(x))^2), \quad \mathbf{k_2}(x) = f(x)^2\left(k_2 + \frac{1}{4}(V_0f(x))^2\right).$$

In particular, the metric measure space (M,d,μ) *satisfies the condition* $\mathcal{MCP}(k_1,k_2;2n,2n+1)$.

Corollary 2.1 is sharp in the sense that all the inequalities in Corollary 2.1, including both assumptions and conclusions, are equality in the case of the Heisenberg group and the complex Hopf fibration. More precisely, the Heisenberg group is the sub-Riemannian manifold where M is the $2n+1$-dimensional Euclidean space with coordinates

$x_1, ..., x_n, y_1, ..., y_n, z$. The distribution $\mathcal{D}$ is given by the span of the vector fields

$$\left\{\partial_{x_i} - \frac{1}{2}y_i\partial_z, \partial_{y_i} + \frac{1}{2}x_i\partial_z \Big| i = 1, ..., n\right\}$$

and the sub-Riemannian metric is defined in such a way that this family of vector fields is orthonormal. The symmetry V_0 is given by ∂_z and the measure μ coincides with the $2n+1$-dimensional Lebesgue measure $\mathcal{L}^{2n+1}$.

Corollary 2.2. *The Heisenberg group satisfies*

$$\mathcal{L}^{2n+1}(\varphi_t(U)) = \int_U \frac{(1-t)\mathcal{M}_1(\mathbf{k_1}(x), t)\mathcal{M}_2^{2n-2}(\mathbf{k_2}(x), t)}{\mathcal{M}_1(\mathbf{k_1}(x), 0)\mathcal{M}_2^{2n-2}(\mathbf{k_2}(x), 0)} d\mathcal{L}^{2n+1}(x)$$

for any Borel set U, where

$$\mathbf{k_1}(x) = f(x)^2(V_0 f(x))^2, \quad \mathbf{k_2}(x) = \frac{f(x)^2(V_0 f(x))^2}{4}.$$

In particular, the metric measure space $(\mathbb{H}, d, \mathcal{L}^{2n+1})$ satisfies the condition $\mathcal{MCP}(0, 0; 2n, 2n+1) = \mathcal{MCP}(0, 2n+3)$.

For the complex Hopf fibration, the manifold M is the $2n+1$ dimensional sphere $M = \{(z_1, ..., z_{n+1}) \in \mathbb{C}^{n+1} | |z_1|^2 + ... + |z_{n+1}|^2 = 1\}$. The symmetry V_0 is the infinitesimal generator of the action $z \mapsto e^{2\pi i t}z$ and the distribution $\mathcal{D}$ is the orthogonal complement of V_0 with respect to the round metric on M.

Corollary 2.3. *The complex Hopf fibration satisfies*

$$\mu(\varphi_t(U)) = \int_U \frac{(1-t)\mathcal{M}_1(\mathbf{k_1}(x), t)\mathcal{M}_2^{2n-2}(\mathbf{k_2}(x), t)}{\mathcal{M}_1(\mathbf{k_1}(x), 0)\mathcal{M}_2^{2n-2}(\mathbf{k_2}(x), 0)} d\mu(x)$$

for any Borel set U, where

$$\mathbf{k_1}(x) = f(x)^2(4 + (V_0 f(x))^2), \quad \mathbf{k_2}(x) = \frac{f(x)^2}{4}\left(4 + (V_0 f(x))^2\right).$$

In particular, the metric measure space (M, d, μ) satisfies the condition $\mathcal{MCP}(4, 1; 2n, 2n+1)$ and hence $\mathcal{MCP}(0, 2n+3)$.

Finally, we state the corresponding Bonnet-Myer's type Theorem in this setting.

Theorem 2.2. *Assume that the tensor J is parallel and the Riemann curvature tensor* $\mathbf{Rm}$ *satisfies*

$$\langle \mathbf{Rm}(Jv, v)v, Jv \rangle \geq k_1 |v|^4$$

for some positive constant k_1. Then the diameter of the manifold M with respect to the corresponding sub-Riemannian metric is less than or equal to $\frac{2\pi}{\sqrt{k_1}}$.

On the other hand, if $\lambda_3 + 2\lambda_1^2 \leq 0$ and

$$\mathbf{Rc}(v,v) - \frac{1}{|Jv|^2} \langle \mathbf{Rm}(Jv,v)v, Jv\rangle \geq (2n-2)k_2|v|^2$$

for some positive constant k_2, then the diameter of the manifold M with respect to the corresponding sub-Riemannian metric is less than or equal to $\frac{\pi}{\sqrt{k_2}}$.

The rest of the sections will be devoted to the proof of the main results.

3. Sub-Riemannian Geodesic Flows and Measure Contraction

In this section, we recall the definition of the sub-Riemannian geodesic flow and its connections with the contraction of measures appeared in [1, 2, 12].

As in the Riemannian case, the (constant speed) minimizers of (2.1) can be found by minimizing the following kinetic energy functional

$$\inf_{\gamma\in\Gamma} \int_0^1 \frac{1}{2}|\dot\gamma(t)|^2 dt. \tag{3.2}$$

In the Riemannian case, the minimizers of (3.2) are given by the geodesic equation, the Euler-Lagrange equation of the functional (3.2). In the sub-Riemannian case, the minimization problem in (3.2) becomes a constrained minimization problem and it is more convenient to look at the geodesic flow from the Hamiltonian point of view in this case. For this, let $\mathbf{H} : T^*M \to \mathbb{R}$ be the Hamiltonian defined by the Legendre transform:

$$\mathbf{H}(\mathbf{x},\mathbf{p}) = \sup_{\mathbf{v}\in\mathcal{D}} \left(\mathbf{p}(\mathbf{v}) - \frac{1}{2}|\mathbf{v}|^2\right).$$

This Hamiltonian, in turn, defines a Hamiltonian vector field $\vec{\mathbf{H}}$ on the cotangent bundle T^*M which is a sub-Riemannian analogue of the geodesic equation. It is given, in the local coordinates $(x_1, ..., x_n, p_1, ..., p_n)$, by

$$\vec{\mathbf{H}} = \sum_{i=1}^n \left(\mathbf{H}_{p_i}\partial_{x_i} - \mathbf{H}_{x_i}\partial_{p_i}\right).$$

We assume, through out this paper, that the vector field $\vec{\mathbf{H}}$ defines a complete flow which is denoted by $e^{t\vec{\mathbf{H}}}$. In the Riemannian case, the minimizers of (3.2) are given by the projection of the trajectories of $e^{t\vec{\mathbf{H}}}$ to the manifold M. In the sub-Riemannian case, minimizers obtained this way are called normal geodesics and they do not give all the minimizers of (3.2) in general (see [18] for more detailed discussions on this). On the other hand, all minimizers of (3.2) are normal if the distribution $\mathcal{D}$ is contact (see [18]).

Next, we discuss an analogue of the Jacobi equation in the above Hamiltonian setting. For this, let ω be the canonical symplectic form of the cotangent bundle T^*M. In local coordinates $(x_1, ..., x_n, p_1, ..., p_n)$, ω is given by

$$\omega = \sum_{i=1}^{n} dx_i \wedge dp_i.$$

Let $\pi : T^*M \to M$ be the canonical projection and let **ver** the vertical sub-bundle of the cotangent bundle T^*M defined by

$$\mathbf{ver}_{(\mathbf{x},\mathbf{p})} = \{v \in T_{(\mathbf{x},\mathbf{p})}T^*M | \pi_*(v) = 0\}.$$

Recall that a n-dimensional subspace of a symplectic vector space is Lagrangian if the symplectic form vanishes when restricted to the subspace. Each vertical space $\mathbf{ver}_{(\mathbf{x},\mathbf{p})}$ is a Lagrangian subspace of the symplectic vector space $T_{(\mathbf{x},\mathbf{p})}T^*M$. Since the flow $e^{t\vec{\mathbf{H}}}$ preserves the symplectic form ω, it also sends a Lagrangian subspace to another Lagrangian one. Therefore, the following also forms a one-parameter family of Lagrangian subspaces contained in $T_{(\mathbf{x},\mathbf{p})}T^*M$

$$\mathfrak{J}_{(\mathbf{x},\mathbf{p})}(t) = e_*^{-t\vec{\mathbf{H}}}(\mathbf{ver}_{e^{t\vec{\mathbf{H}}}(\mathbf{x},\mathbf{p})}). \tag{3.3}$$

This family defines a curve in the Lagrangian Grassmannian (the space of Lagrangian subspaces) of $T_{(\mathbf{x},\mathbf{p})}T^*M$ and it is called the Jacobi curve at $(\mathbf{x}, \mathbf{p})$ of the flow $e^{t\vec{\mathbf{H}}}$.

Assume that the distribution is contact. Then we have the following particular case of the results in [14, 15].

Theorem 3.3. *Assume that the distribution $\mathcal{D}$ is contact. Then there exists a one-parameter family of bases*

$$E(t) = (E_a(t), E_b(t), E_{c_1}(t), ..., E_{c_{2n-1}}(t))^T$$
$$F(t) = (F_a(t), F_b(t), F_{c_1}(t), ..., F_{c_{2n-1}}(t))^T$$

of the symplectic vector space $T_{(\mathbf{x},\mathbf{p})}T^*M$ such that the followings hold for any t:

1. $\mathfrak{J}_{(\mathbf{x},\mathbf{p})}(t) = \mathrm{span}\{E_a(t), E_b(t), E_{c_1}(t), ..., E_{c_{2n-1}}(t)\}$,
2. $\mathrm{span}\{F_a(t), F_b(t), F_{c_1}(t), ..., F_{c_{2n-1}}(t)\}$ *is a family of Lagrangian subspaces,*

(3) $\omega(E_a(t), F_a(t)) = \omega(E_b(t), F_b(t)) = 1$,
(4) $\omega(E_{c_i}(t), F_{c_j}(t)) = \delta_{ij}$,
(5) $\dot{E}(t) = C_1 E(t) + C_2 F(t)$,
(6) $\dot{F}(t) = -R(t)E(t) - C_1^T F(t)$,

where $R(t)$ is a symmetric matrix, C_1 and C_2 are $(2n+1)\times(2n+1)$ matrices defined by

(1) $\tilde{C}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *is a 2×2 matrix,*
(2) $\tilde{C}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *is a 2×2 matrix,*
(3) $C_1 = \begin{pmatrix} \tilde{C}_1 & O \\ O & O \end{pmatrix}$
(4) $C_2 = \begin{pmatrix} \tilde{C}_2 & O \\ O & I \end{pmatrix}$.

Moreover, a moving frame

$$\tilde{E}_a(t), \tilde{E}_b(t), \tilde{E}_{c_1}(t), ..., \tilde{E}_{c_{2n-1}}(t), \tilde{F}_a(t), \tilde{F}_b(t), \tilde{F}_{c_1}(t), ..., \tilde{F}_{c_{2n-1}}(t)$$

satisfies conditions (1)-(6) above if and only if

$$(\widetilde{E}_a(t), \widetilde{E}_b(t), \widetilde{F}_a(t), \widetilde{F}_b(t)) = \pm(E_a(t), E_b(t), F_a(t), F_b(t)) \tag{3.4}$$

and there exists a constant orthogonal matrix U of size $(2n-1)\times(2n-1)$ such that for all t

$$\widetilde{E}_{c_i}(t) = \sum_{j=1}^{2n-1} U_{ij} E_{c_j}(t) \text{ and } \widetilde{F}_{c_i}(t) = \sum_{j=1}^{2n-1} U_{ij} F_{c_j}(t). \tag{3.5}$$

We call any frame $(E(t), F(t))$ in Theorem 3.3 a canonical frame at the point $(\mathbf{x}, \mathbf{p})$ and call the equations in (5) and (6) of Theorem 3.3 the structural equation of the Jacobi curve (3.3). Note that the conditions (3) - (4) means that the canonical frame is a family of symplectic bases.

Let us fix a point $\mathbf{x}_0$ in M and let $\mathfrak{f}(\mathbf{x}) = -\frac{1}{2}d^2(\mathbf{x}, \mathbf{x}_0)$. By the result of [6], $\mathfrak{f}$ is locally semi-concave in $M - \{x_0\}$. In particular, it is differentiable almost everywhere and we can define the family of Borel maps $\varphi_t : M \to M$ by $\varphi_t(\mathbf{x}) = \pi(e^{t\vec{\mathbf{H}}}(d\mathfrak{f}_{\mathbf{x}}))$, where $0 \le t \le 1$. Note that $t \mapsto \varphi_t(\mathbf{x})$ is a minimizing geodesic between the points $\mathbf{x}$ and $\mathbf{x}_0$ (see for instance [1]).

Let M be a contact sub-Riemannian manifold with symmetry. This means that the sub-Riemannian structure on M is defined by a contact distribution $\mathcal{D}$ and there is a sub-Riemannian isometry V_0 which is transversal to $\mathcal{D}$. We can extend the sub-Riemannian on M to a Riemannian one by the conditions that the vector field V_0 is of unit length and is orthogonal to $\mathcal{D}$. The corresponding Riemannian measure is denoted by μ. By the result in [9], the measures $(\varphi_t)_*\mu$ are absolutely continuous with respect to μ for all time t in the interval $[0,1)$. If $(\varphi_t)_*\mu = \rho_t\mu$, then the following equation holds on a set of full measure where $\mathfrak{f}$ is twice differentiable:

$$\rho_t(\varphi_t(\mathbf{x}))\det((d\varphi_t)_{\mathbf{x}}) = 1$$

and the determinant is computed with respect to frames of the above mentioned Riemannian structure. Moreover, the map φ_t is invertible for all t in $[0,1)$ and so we have

$$\mu(\varphi_t(U)) = \int_U \frac{1}{\rho_t(\varphi_t(\mathbf{x}))} d\mu(\mathbf{x}) = \int_U \det((d\varphi_t)_{\mathbf{x}}) d\mu(\mathbf{x}). \tag{3.6}$$

Therefore, in order to prove the main results and the measure contraction properties, it remains to estimate $\det((d\varphi_t)_{\mathbf{x}})$ which can be done using the canonical frame mentioned above. The explanations on this will occupy the rest of this section.

Let $\mathbf{x}$ be a point where the function $\mathfrak{f}$ is twice differentiable and let $(E(t), F(t))$ be a canonical frame at the point $(\mathbf{x}, d\mathfrak{f}_{\mathbf{x}})$. Let

$$\varsigma_a = \pi_*(F_a(0)), \quad \varsigma_b = \pi_*(F_b(0)), \quad \varsigma_{c_i} = \pi_*(F_{c_i}(0)),$$

be the projection of the frame $F(0)$ onto the tangent bundle TM. Let $dd\mathfrak{f}$ be the differential of the map $\mathbf{x} \mapsto d\mathfrak{f}_{\mathbf{x}}$ which pushes the above frame on $T_{\mathbf{x}}M$ to a frame on $T_{(\mathbf{x},d\mathfrak{f})}T^*M$. Therefore, we can let $A(t)$ and $B(t)$ be the matrices defined by

$$dd\mathfrak{f}(\varsigma) = A(t)E(t) + B(t)F(t), \tag{3.7}$$

where $\varsigma = \left(\varsigma_a, \varsigma_b, \varsigma_{c_1}, \cdots, \varsigma_{c_{2n-1}}\right)^T$ and $dd\mathfrak{f}(\varsigma)$ is the column obtained by applying $dd\mathfrak{f}$ to each entries of ς.

Lemma 3.1. *Let $S(t) = B(t)^{-1}A(t)$. Then*

$$\mu(\varphi_t(U)) \geq \frac{1}{\lambda_1\lambda_2}\int_U e^{-\int_0^t \mathbf{tr}(S(\tau)C_2)d\tau} d\mu(\mathbf{x}),$$

where λ_1 and λ_2 are the operator norms of J and J^{-1}, respectively.

Moreover, $S(t)$ satisfies the following matrix Riccati equation

$$\dot{S}(t) - S(t)C_2S(t) + C_1^TS(t) + S(t)C_1 - R(t) = 0, \quad \lim_{t\to 1} S(t)^{-1} = 0.$$

By Lemma 3.1, it remains to investigate the Riccati equation satisfied by $S(t)$ and the curvature matrix $R(t)$ which will be done in next two sections.

Proof of Lemma 3.1. By (3.7) and the definition of φ_t, we have

$$d\varphi_t(\varsigma) = B(t)(\pi_* de^{t\vec{\mathbf{H}}}F(t)).$$

Note that $\tau \mapsto de^{t\vec{\mathbf{H}}}F(t+\tau)$ is a canonical frame at $e^{t\vec{\mathbf{H}}}(\mathbf{x}, d\mathfrak{f})$. Therefore, by Lemma 6.4, we have

$$\begin{aligned}\lambda_1|\nabla_{\mathbf{hor}}\mathfrak{f}(\mathbf{x})||\det(d\varphi_t)| &\geq |\mu(d\varphi_t(\varsigma))| \\ = |\det(B(t))\mu(d\pi de^{t\vec{\mathbf{H}}}F(t))| &\geq \frac{1}{\lambda_2}|\det(B(t))||\nabla_{\mathbf{hor}}\mathfrak{f}(\mathbf{x})|\end{aligned}$$

where λ_1 and λ_2 are the operator norms of J and J^{-1}, respectively. Here $\nabla_{\mathbf{hor}}\mathfrak{f}$ denotes the horizontal gradient of $\mathfrak{f}$ defined by $d\mathfrak{f}(v) = \langle \nabla_{\mathbf{hor}}\mathfrak{f}, v\rangle$, where v is any vector in the distribution $\mathcal{D}$.

By combining this with (6.4), we obtain

$$\mu(\varphi_t(U)) \geq \frac{1}{\lambda_1\lambda_2}\int_U |\det(B(t))| d\mu(\mathbf{x}). \tag{3.8}$$

On the other hand, by differentiating (3.7) with respect to time t and using the structural equation, we obtain

$$\dot{A}(t) + A(t)C_1 - B(t)R_t = 0, \quad \dot{B}(t) + A(t)C_2 - B(t)C_1^T = 0. \tag{3.9}$$

Therefore,

$$\frac{d}{dt}\det(B(t)) = \det(B(t))\mathbf{tr}(B(t)^{-1}\dot{B}(t)) = -\det(B(t))\mathbf{tr}(S(t)C_2).$$

By setting $t = 0$ and apply π_* on each side of (3.7), we have $B(0) = I$. Therefore, we obtain

$$\det(B(t)) = e^{-\int_0^t \mathbf{tr}(S(\tau)C_2)d\tau}.$$

By combining this with (3.8), we obtain the first assertion.

Since $\varphi_1(\mathbf{x}) = \mathbf{x}_0$ for all $\mathbf{x}$, we have $d\varphi_1 = 0$ and so $B(1) = 0$. By (3.9) and the definition of $S(t)$, we have

$$\dot{S}(t) - S(t)C_2S(t) + C_1^TS(t) + S(t)C_1 - R(t) = 0, \quad \lim_{t\to 1} S(t)^{-1} = 0$$

as claimed. □

4. On the matrix Riccati equation

According to Lemma 3.1, we need to estimate the term $\mathbf{tr}(S(t)C_2)$. In this section, we provide two such estimates which lead to the main results.

Throughout this section, we assume that the matrix $R(t)$ is of the form

$$R(t) = \left(\begin{array}{cccc} 0 & 0 & O_{2n-2} & 0 \\ 0 & R_{bb}(t) & R_{cb}(t) & 0 \\ O_{2n-2}^T & R_{cb}(t)^T & R_{cc}(t) & O_{2n-2}^T \\ 0 & 0 & O_{2n-2} & 0 \end{array} \right),$$

where O_{2n-2} is the zero matrix of size $1 \times (2n-2)$. We will see the reasons for this choice in section 5.

The following is a consequence of the result in [21].

Lemma 4.2. *Assume that the curvature $R(t)$ satisfies*

$$\left(\begin{array}{cc} R_{bb}(t) & R_{cb}(t) \\ R_{cb}(t)^T & R_{cc}(t) \end{array} \right) \geq \left(\begin{array}{cc} \mathfrak{k}_1 & 0 \\ 0 & \mathfrak{k}_2 I \end{array} \right),$$

where I is of size $(2n-2)\times(2n-2)$, $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are two constants. Then

$$e^{-\int_0^t \boldsymbol{tr}(C_2 S(\tau))d\tau} \geq (1-t)^{2n+3} \frac{\mathcal{M}_1(\mathfrak{k}_1, t)\mathcal{M}_2^{2n-2}(\mathfrak{k}_2, t)}{\mathcal{M}_1(\mathfrak{k}_1, 0)\mathcal{M}_2^{2n-2}(\mathfrak{k}_2, 0)},$$

where

$$\mathcal{D}(k,t) = \sqrt{|k|}(1-t),$$

$$\mathcal{M}_1(k,t) = \begin{cases} \frac{2-2\cos(\mathcal{D}(k,t)) - \mathcal{D}(k,t)\sin(\mathcal{D}(k,t))}{\mathcal{D}(k,t)^4} & \text{if } k > 0 \\ \frac{1}{12} & \text{if } k = 0 \\ \frac{2-2\cosh(\mathcal{D}(k,t)) + \mathcal{D}(k,t)\sinh(\mathcal{D}(k,t))}{\mathcal{D}(k,t)^4} & \text{if } k < 0, \end{cases}$$

$$\mathcal{M}_2(k,t) = \begin{cases} \frac{\sin(\mathcal{D}(k,t))}{\mathcal{D}(k,t)} & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ \frac{\sinh(\mathcal{D}(k,t))}{\mathcal{D}(k,t)} & \text{if } k < 0. \end{cases}$$

Moreover, equality holds if

$$\left(\begin{array}{cc} R_{bb}(t) & R_{cb}(t) \\ R_{cb}(t)^T & R_{cc}(t) \end{array} \right) = \left(\begin{array}{cc} \mathfrak{k}_1 & 0 \\ 0 & \mathfrak{k}_2 I \end{array} \right),$$

Proof. We only prove the case when both constants $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are positive. The proofs for other cases are similar and are therefore omitted. Recall that $S(t)$ satisfies

$$\dot{S}(t) - S(t)C_2S(t) + C_1^T S(t) + S(t)C_1 - R(t) = 0, \quad \lim_{t\to 1} S(t)^{-1} = 0.$$

Let $\Gamma(t)$ be a solution of the following

$$\dot{\Gamma}(t) - \Gamma(t)C_2\Gamma(t) + C_1^T\Gamma(t) + \Gamma(t)C_1 - K = 0, \quad \lim_{t\to 1} \Gamma(t)^{-1} = 0,$$

where $K = \left(\begin{array}{cc} \mathfrak{k}_1 & 0 \\ 0 & \mathfrak{k}_2 I \end{array} \right)$.

By the result of [21], $S(t) \le \Gamma(t)$. Therefore, $\mathbf{tr}(C_2S(t)) \le \mathbf{tr}(C_2\Gamma(t))$. On the other hand, $\Gamma(t)$ can be computed using the result in [12] and it follows that

$$\mathbf{tr}(C_2S(t)) \le \frac{\sqrt{\mathfrak{k}_1}(\sin(\mathcal{D}(\mathfrak{k}_1,t)) - \mathcal{D}(\mathfrak{k}_1,t)\cos(\mathcal{D}(\mathfrak{k}_1,t)))}{2-2\cos(\mathcal{D}(\mathfrak{k}_1,t)) - \mathcal{D}(\mathfrak{k}_1,t)\sin(\mathcal{D}(\mathfrak{k}_1,t))} + (2n-2)\sqrt{\mathfrak{k}_2}\cot(\mathcal{D}(\mathfrak{k}_2,t)) + \frac{1}{1-t}.$$

The assertion follows from integrating the above inequality. ☐

Next, we consider the case where the assumptions are weaker than those in Lemma 4.2.

Lemma 4.3. *Assume that the curvature $R(t)$ satisfies $R_{bb}(t) \ge \mathfrak{k}_1$ and $\boldsymbol{tr}(R_{cc}(t)) \ge \mathfrak{k}_2(2n-2)$ for some constants $\mathfrak{k}_1$ and $\mathfrak{k}_2$. Then*

$$e^{-\int_0^t \boldsymbol{tr}(C_2S(\tau))d\tau} \ge (1-t)^{2n+3}\frac{\mathcal{M}_1(\mathfrak{k}_1,t)\mathcal{M}_2^{2n-2}(\mathfrak{k}_2,t)}{\mathcal{M}_1(\mathfrak{k}_1,0)\mathcal{M}_2^{2n-2}(\mathfrak{k}_2,0)}.$$

Proof. Once again, we only prove the case when both constants $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are positive. Let us write

$$S(t) = \begin{pmatrix} S_1(t) & S_2(t) & S_3(t) \\ S_2(t)^T & S_4(t) & S_5(t) \\ S_3(t)^T & S_5(t)^T & S_6(t) \end{pmatrix},$$

where $S_1(t)$ is a 2×2 matrix and $S_6(t)$ is 1×1. Then

$$\begin{aligned} &\dot{S}_1(t) - S_1(t)\tilde{C}_2S_1(t) - S_2(t)S_2(t)^T \\ &- S_3(t)^2 + \tilde{C}_1^TS_1(t) + S_1(t)\tilde{C}_1 - R_1(t) = 0, \\ &\dot{S}_4(t) - S_4(t)^2 - S_5(t)S_5(t)^T - S_2(t)^T\tilde{C}_2S_2(t) - R_{cc}(t) = 0, \\ &\dot{S}_6(t) - S_6(t)^2 - S_5(t)^TS_5(t) - S_3(t)^T\tilde{C}_2S_3(t) = 0, \end{aligned} \tag{4.10}$$

where $\tilde{C}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\tilde{C}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $R_1(t) = \begin{pmatrix} 0 & 0 \\ 0 & R_{bb}(t) \end{pmatrix}$.

By the same argument as in Lemma 4.2, we have

$$\mathbf{tr}(\tilde{C}_2S_1(t)) \le \frac{\sqrt{\mathfrak{k}_1}(\sin(\mathcal{D}(\mathfrak{k}_1,t)) - \mathcal{D}(\mathfrak{k}_1,t)\cos(\mathcal{D}(\mathfrak{k}_1,t)))}{2-2\cos(\mathcal{D}(\mathfrak{k}_1,t)) - \mathcal{D}(\mathfrak{k}_1,t)\sin(\mathcal{D}(\mathfrak{k}_1,t))}. \tag{4.11}$$

For the term $S_4(t)$, we can take the trace and obtain

$$\frac{d}{dt}\mathbf{tr}(S_4(t)) \ge \frac{1}{2n-2}\mathbf{tr}(S_4(t))^2 + (2n-2)\mathfrak{k}_2.$$

Therefore, an argument as in Lemma 4.2 again gives

$$\mathbf{tr}S_4(t) \le \sqrt{|\mathfrak{k}_2|}(2n-2)\cot\left(\mathcal{D}(\mathfrak{k}_2,t)\right). \tag{4.12}$$

Finally, for the term $S_6(t)$, we also have

$$\dot{S}_6(t) \geq S_6(t)^2.$$

Therefore,

$$S_6(t) \leq \frac{1}{1-t}.$$

By combining this with (4.11) and (4.12), we obtain

$$\begin{aligned}\mathbf{tr}(C_2 S(t)) \leq \sqrt{|\mathfrak{k}_2|}(2n-2)\cot\left(\mathcal{D}(\mathfrak{k}_2, t)\right) + \frac{1}{1-t} \\ + \frac{\sqrt{\mathfrak{k}_1}(\sin(\mathcal{D}(\mathfrak{k}_1, t)) - \mathcal{D}(\mathfrak{k}_1, t)\cos(\mathcal{D}(\mathfrak{k}_1, t)))}{2 - 2\cos(\mathcal{D}(\mathfrak{k}_1, t)) - \mathcal{D}(\mathfrak{k}_1, t)\sin(\mathcal{D}(\mathfrak{k}_1, t))}.\end{aligned}$$

The rest follows as in Lemma 4.2. □

5. Curvature of Contact Sub-Riemannian manifolds with symmetry

Let $(E(t), F(t))$ be a canonical frame at a point $(\mathbf{x}, \mathbf{p})$ of the cotangent bundle T^*M. Recall that the vertical bundle $\mathbf{ver}$ of TT^*M is given by

$$\mathbf{ver} = \{V \in TT^*M | \pi_* V = 0\}.$$

The linear map $\mathfrak{R} : \mathbf{ver} \to \mathbf{ver}$ having the matrix $R(0)$ with respect to the canonical frame $(E(0), F(0))$ is called the curvature map. More precisely,

$$\mathfrak{R}_{(\mathbf{x},\mathbf{p})}(V) = AR(0)E(0),$$

where $V = AE(0)$ and A is any row vectors of suitable size. Moreover it follows from Theorem 3.3 that the above definition does not depend on the choice of the canonical frame. In this section, we discuss $\mathfrak{R}$ in the case where the manifold M is a contact sub-Riemannian manifold with symmetry.

Let α be a contact form of the given distribution $\mathcal{D}$. Then the restriction of $d\alpha$ onto the distribution $\mathcal{D}$ is a non-degenerate 2-form. This defines a skew-symmetric linear bundle map $J : \mathcal{D} \to \mathcal{D}$ by

$$d\alpha(X, Y) = \langle JX, Y \rangle$$

for any pair of vectors X and Y contained in $\mathcal{D}$. In addition assume that the Reeb field V_0 is an infinitesimal symmetry, i.e.

$$e_*^{tV_0}\mathcal{D} = \mathcal{D}\ , \quad (e^{tV_0})^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle\ .$$

Assume also that V_0 is transversal to the distribution $\mathcal{D}$.

We also assume that the quotient $\widetilde{M}$ of the manifold M by the symmetry V_0 is also a manifold. The quotient map $\pi_0 : M \to \widetilde{M}$ defines an identification of $\mathcal{D}$ with $T\widetilde{M}$. Therefore, both the sub-Riemannian metric $\langle\cdot,\cdot\rangle$ and the map J descend to $T\widetilde{M}$ which are denoted by the same symbol. We also let $u_0 : T^*M \to \mathbb{R}$ be the function defined by $u_0(\mathbf{x},\mathbf{p}) = \mathbf{p}(V_0(\mathbf{x}))$.

Next, we introduce some notations for later use. We recall the identification of the cotangent space $T^*_{\mathbf{x}}M$ and the vertical space $\mathbf{ver}_{(\mathbf{x},\mathbf{p})}$ by

$$\mathbf{q} \mapsto \mathbf{q}^{\mathbf{ver}} := \frac{d}{dt}(\mathbf{p}+t\mathbf{q})\Big|_{t=0},$$

where $\mathbf{p}$ and $\mathbf{q}$ are covectors in $T^*_{\mathbf{x}}M$. By using the above identification, we can assign a unique covector $\mathbf{q}$ in $T^*_{\mathbf{x}}M$ to each vector $\mathbf{v}$ in the vertical bundle $\mathbf{ver}_{(\mathbf{x},\mathbf{p})}$ such that $\mathbf{q}^{\mathbf{ver}} = \mathbf{v}$. This, in turn, defines a vector $\tilde{\mathbf{v}}$ in the distribution $\mathcal{D}$ by $\mathbf{q}(w) = \langle\tilde{\mathbf{v}}, w\rangle$, where w is any vector in the distribution $\mathcal{D}$. Finally, the vector $\mathbf{v}^h$ in $T\widetilde{M}$ is defined by $\mathbf{v}^h = (\pi_0)_*\tilde{\mathbf{v}}$. We will also denote this vector by $\mathbf{q}^h$.

The linear map $\mathbf{q} \mapsto I(\mathbf{q}) := \mathbf{q}^h = \mathbf{v}^h$ is surjective with a 1-dimensional kernel. Therefore, given any vector X in $T_{\pi_0(\mathbf{x})}\widetilde{M}$, there is a 1-dimensional affine subspace of the cotangent space $T^*_{\mathbf{x}}M$ such that any covector $\mathbf{q}$ inside satisfies $I(\mathbf{q}) = \mathbf{v}^h$. Moreover, there is a unique covector $\mathbf{q}_0$ in this affine space which satisfies the condition $\mathbf{q}_0(V_0) = 0$. Here V_0 is the symmetry introduced earlier. Finally, we denote by X^v the vector $\mathbf{q}_0^{\mathbf{ver}}$ in the vertical space $T_{(\mathbf{x},\mathbf{p})}T^*M$.

The frame $E(0) = (E_a(0), E_b(0), E_{c_1}(0), ..., E_{c_{2n-1}}(0))^T$ defines a splitting of the vertical space $\mathbf{ver} = \mathbf{ver}_a \oplus \mathbf{ver}_b \oplus \mathbf{ver}_c$, which are characterized as follows (see [15, Section 3] for a proof). Let ∂_{u_0} be the vector field in $\mathbf{ver}_a$ satisfying the condition $du_0(\partial_{u_0}) = 1$.

Proposition 5.1. *The subspaces* $\mathbf{ver}_a, \mathbf{ver}_b$ *and* $\mathbf{ver}_c$ *are given by the followings:*

(1) $\mathbf{ver}_a := span\{E_a(0)\} = span\{\mathcal{E}_a := \frac{\partial_{u_0}}{|J\mathbf{p}^h|}\}$,
(2) $\mathbf{ver}_b := span\{E_b(0)\} = span\left\{\mathcal{E}_b := \frac{(J\mathbf{p}^h)^v}{|J\mathbf{p}^h|} + \vec{H}\left(\frac{1}{|J\mathbf{p}^h|}\right)\partial_{u_0}\right\}$,
(3) $\mathbf{ver}_c := span\{E_{c_1}(0), ..., E_{c_{2n-1}}(0)\}$
$= \{X^v + \mathcal{A}(X^v)\frac{\partial_{u_0}}{|J\mathbf{p}^h|} | X \in span\{J\mathbf{p}^h\}^\perp\}$,

where $\mathcal{A}$ *is a linear functional on* $\mathbf{ver}_c$ *defined by* $\mathcal{A}(\mathbf{v}) = -\left\langle \mathbf{v}^h, \mathfrak{V}_1^h\right\rangle$ *and* $\mathfrak{V}_1 \in \mathbf{ver}_c$ *such that*

$$\mathfrak{V}_1^h = -\frac{2}{|J\mathbf{p}^h|}\nabla_{\mathbf{p}^h}J(\mathbf{p}^h) + \frac{u_0}{|J\mathbf{p}^h|}J^2\mathbf{p}^h + \frac{u_0|J\mathbf{p}^h|}{|\mathbf{p}^h|^2}\mathbf{p}^h + \frac{2}{|J\mathbf{p}^h|^3}\left\langle\nabla_{\mathbf{p}^h}J(\mathbf{p}^h), J\mathbf{p}^h\right\rangle J\mathbf{p}^h.$$

Let V be a vector in $\mathbf{ver}_{z_1}$, where $z_1 = a, b, c_1, ..., c_{2n-1}$. The $\mathbf{ver}_{z_2}$-component of $\mathfrak{R}(V)$ is denoted by $\mathfrak{R}(z_1, z_2)(V)$. In other words,

$$\mathfrak{R}(V) = \mathfrak{R}(z_1, a)(V) + \mathfrak{R}(z_1, b)(V) + \mathfrak{R}(z_1, c)(V),$$

where $\mathfrak{R}(z_1, a)(V)$, $\mathfrak{R}(z_1, b)(V)$, and $\mathfrak{R}(z_1, c)(V)$ are contained in $\mathbf{ver}_a$, $\mathbf{ver}_b$, and $\mathbf{ver}_c$, respectively. The following theorems follows from [15] (see Appendix I for details).

Theorem 5.4. *The components $\mathfrak{R}(c, c)$ and $\mathfrak{R}(b, c)$ of the curvature $\mathfrak{R}$ satisfy the followings:*

(1) $\left\langle \left(\mathfrak{R}(c, c)(\mathbf{v})\right)^h, \mathbf{v}^h \right\rangle = \left\langle \mathbf{Rm}(\mathbf{v}^h, \mathbf{p}^h)\mathbf{p}^h, \mathbf{v}^h \right\rangle$
$+ u_0(\mathbf{x}, \mathbf{p}) \left\langle \mathbf{v}^h, \nabla_{\mathbf{v}^h} J(\mathbf{p}^h) \right\rangle + \frac{u_0^2(\mathbf{x},\mathbf{p})}{4} |J\mathbf{v}^h|^2 - \frac{1}{4}\mathcal{A}(\mathbf{v})^2$,

(2) $\mathfrak{R}(c, b)\mathbf{v} = \rho(c, b)(\mathbf{v})\mathcal{E}_b$,

(3) $\rho(c, b)(\mathbf{v}) = \frac{1}{|J\mathbf{p}^h|} \left\langle \mathbf{Rm}(J\mathbf{p}^h, \mathbf{p}^h)\mathbf{p}^h, \mathbf{v}^h \right\rangle - \frac{3}{|J\mathbf{p}^h|} \left\langle \mathbf{v}^h, \nabla^2 J(\mathbf{p}^h, \mathbf{p}^h, \mathbf{p}^h) \right\rangle$
$+ \frac{4u_0(\mathbf{x},\mathbf{p})}{|J\mathbf{p}^h|} \left\langle \mathbf{v}^h, \nabla_{\mathbf{p}^h} J(J\mathbf{p}^h) + \nabla_{J\mathbf{p}^h} J(\mathbf{p}^h) \right\rangle + \frac{u_0^2(\mathbf{x},\mathbf{p})}{|J\mathbf{p}^h|} \left\langle J\mathbf{v}^h, J^2\mathbf{p}^h \right\rangle$
$+ \frac{8}{|J\mathbf{p}^h|^3} \left\langle J\mathbf{p}^h, \nabla_{\mathbf{p}^h} J(\mathbf{p}^h) \right\rangle \left\langle \mathbf{v}^h, \nabla_{\mathbf{p}^h} J(\mathbf{p}^h) \right\rangle$
$- \frac{4u_0(\mathbf{x},\mathbf{p})}{|J\mathbf{p}^h|^3} \left\langle J\mathbf{p}^h, \nabla_{\mathbf{p}^h} J(\mathbf{p}^h) \right\rangle \left\langle \mathbf{v}^h, J^2\mathbf{p}^h \right\rangle$.

Remark 5.1. *In Theorem 5.4, we use the following definition of the Riemann curvature tensor*

$$\mathbf{Rm}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

This is different from that of [15] *by a minus sign.*

Theorem 5.5. *The component $\mathfrak{R}(b, b)$ of the curvature $\mathfrak{R}$ satisfies the followings:*

(1) $\mathfrak{R}(b, b)\mathcal{E}_b = \rho(b, b)\mathcal{E}_b$,

(2) $\rho(b, b) = \frac{1}{|J\mathbf{p}^h|^2} \left\langle \mathbf{Rm}(\mathbf{p}^h, J\mathbf{p}^h)J\mathbf{p}^h, \mathbf{p}^h \right\rangle - \frac{10}{|J\mathbf{p}^h|^4} \left\langle \nabla_{\mathbf{p}^h} J(\mathbf{p}^h), J\mathbf{p}^h \right\rangle^2$
$+ \frac{6}{|J\mathbf{p}^h|^2} |\nabla_{\mathbf{p}^h} J(\mathbf{p}^h)|^2 + \frac{3}{|J\mathbf{p}^h|^2} \left\langle J\mathbf{p}^h, \nabla^2 J(\mathbf{p}^h, \mathbf{p}^h, \mathbf{p}^h) \right\rangle$
$- \frac{2u_0(\mathbf{x},\mathbf{p})}{|J\mathbf{p}^h|^2} \left\langle J\mathbf{p}^h, \nabla_{J\mathbf{p}^h} J(\mathbf{p}^h) \right\rangle - \frac{3u_0(\mathbf{x},\mathbf{p})}{|J\mathbf{p}^h|^2} \left\langle J\mathbf{p}^h, \nabla_{\mathbf{p}^h} J(J\mathbf{p}^h) \right\rangle$
$- \frac{6u_0(\mathbf{x},\mathbf{p})}{|J\mathbf{p}^h|^2} \left\langle J^2\mathbf{p}^h, \nabla_{\mathbf{p}^h} J(\mathbf{p}^h) \right\rangle + \frac{u_0^2(\mathbf{x},\mathbf{p})}{|J\mathbf{p}^h|^2} |J^2\mathbf{p}^h|^2$,

Let $\mathcal{K}^t$ be the linear maps from $T_{(\mathbf{x},\mathbf{p})}(T^*M)$ to $T_{e^{t\vec{\mathbf{H}}}(\mathbf{x},\mathbf{p})}T^*M$, sending $E(0)$ to $\left(e^{t\vec{\mathbf{H}}}\right)_* E(t)$ and $F^{(\mathbf{x},\mathbf{p})}(0)$ to $\left(e^{t\vec{\mathbf{H}}}\right)_* F(t)$. The map $\mathcal{K}^t$ is called *the parallel transport* along the curve $e^{t\vec{\mathbf{H}}}(\mathbf{x}, \mathbf{p})$ at time t. Note that $\mathcal{K}^t$ sends the vertical space $\mathbf{ver}_{(\mathbf{x},\mathbf{p})}$ to the vertical space $\mathbf{ver}_{e^{t\vec{\mathbf{H}}}(\mathbf{x},\mathbf{p})}$.

Let $S : \mathbf{ver} \to \mathbb{R}$ be a function on the vertical bundle $\mathbf{ver}$. Then the i^{th} derivative of S along a path $t \mapsto \mathcal{K}_t(\mathbf{v})$ is denoted by $S^{(i)}(\mathbf{v})$. More precisely, we have

$$S^{(i)}(\mathbf{v}) = \frac{d^i}{dt^i} S(\mathcal{K}_t(\mathbf{v}))\Big|_{t=0}.$$

In the following theorem, we need this notation for the function $\mathbf{v} \mapsto \mathcal{A}(\mathbf{v})$. The explicit expressions of the derived maps $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}$ (in terms of the tensor J and its covariant derivatives) are given by Proposition 9.3 in Appendix II.

Theorem 5.6. *The curvature maps $\mathfrak{R}(c,a)$ and $\mathfrak{R}(a,a)$ are given by*

(1) $\mathfrak{R}(c,a)\mathbf{v} = \rho(c,a)(\mathbf{v})\frac{\partial_{u_0}}{|J\mathbf{p}^h|}$,

(2) $\rho(c,a)\mathbf{v} = \mathcal{A}^{(2)}(\mathbf{v}) + 2|J\mathbf{p}^h|\vec{\mathbf{H}}\left(\frac{1}{|J\mathbf{p}^h|}\right)\mathcal{A}^{(1)}(\mathbf{v})$
$+|J\mathbf{p}^h|\vec{\mathbf{H}}^2\left(\frac{1}{|J\mathbf{p}^h|}\right)\mathcal{A}(\mathbf{v}) - \left\langle\left(\mathfrak{R}(c,c)\mathbf{v}\right)^h, \mathfrak{V}_1^h\right\rangle + |J\mathbf{p}^h|\vec{\mathbf{H}}\left(\frac{1}{|J\mathbf{p}^h|}\right)\rho(c,b)\mathbf{v}$,

(3) $\mathfrak{R}(a,a)\partial_{u_0} = \rho(a,a)\partial_{u_0}$,

(4) $\rho(a,a) = \vec{\mathbf{H}}\left(\rho(c,b)(\mathfrak{V}_1)\right) + |J\mathbf{p}^h|\vec{\mathbf{H}}\left(\frac{1}{|J\mathbf{p}^h|}\right)\vec{\mathbf{H}}(\rho(b,b))$
$+ \rho(c,a)(\mathfrak{V}_1) - |J\mathbf{p}^h|\vec{\mathbf{H}}\left(\frac{1}{|J\mathbf{p}^h|}\right)\rho(c,b)(\mathfrak{V}_1)$
$+ |J\mathbf{p}^h|\vec{\mathbf{H}}^2\left(\frac{1}{|J\mathbf{p}^h|}\right)\rho(b,b) + |J\mathbf{p}^h|\vec{\mathbf{H}}^4\left(\frac{1}{|J\mathbf{p}^h|}\right).$

Here $\mathfrak{V}_1$ and $\mathfrak{V}_1^h$ are as in Proposition 5.1.

The curvature $\mathfrak{R}$ is much simpler when J is parallel, i.e. $\nabla J = 0$.

Corollary 5.4. *Assume that $\nabla J = 0$. Then*

(1) $\left\langle\left(\mathfrak{R}(c,c)(\mathbf{v})\right)^h, \mathbf{v}^h\right\rangle = \left\langle\mathbf{Rm}(\mathbf{v}^h, \mathbf{p}^h)\mathbf{p}^h, \mathbf{v}^h\right\rangle$
$+ \frac{u_0^2(\mathbf{x},\mathbf{p})}{4}\left(|J\mathbf{v}^h|^2 - \frac{1}{|J\mathbf{p}^h|^2}\left\langle\mathbf{v}^h, J^2\mathbf{p}^h\right\rangle^2\right)$,

(2) $\rho(c,b)\mathbf{v} = \frac{1}{|J\mathbf{p}^h|}\left\langle\mathbf{Rm}(J\mathbf{p}^h, \mathbf{p}^h)\mathbf{p}^h, \mathbf{v}^h\right\rangle + \frac{u_0^2(\mathbf{x},\mathbf{p})}{|J\mathbf{p}^h|}\left\langle J\mathbf{v}^h, J^2\mathbf{p}^h\right\rangle$,

(3) $\rho(b,b) = \frac{1}{|J\mathbf{p}^h|^2}\left\langle\mathbf{Rm}(J\mathbf{p}^h, \mathbf{p}^h)\mathbf{p}^h, J\mathbf{p}^h\right\rangle + \frac{u_0^2(\mathbf{x},\mathbf{p})}{|J\mathbf{p}^h|^2}|J^2\mathbf{p}^h|^2$,

(4) $\mathfrak{R}(c,a) = 0$,

(5) $\mathfrak{R}(a,a) = 0$.

If, in addition to the condition $\nabla J = 0$, we assume that $J^2 = -I$ (i.e. the manifold $\widetilde{M}$ equipped with $(\langle\cdot,\cdot\rangle, J)$ is a Kähler manifold), then

Corollary 5.5. *Assume that $\nabla J = 0$ and $J^2 = -I$. Then*

(1) $\left\langle(\mathfrak{R}(c,c)(\mathbf{v}))^h, \mathbf{v}^h\right\rangle = \left\langle\mathbf{Rm}(\mathbf{v}^h, \mathbf{p}^h)\mathbf{p}^h, \mathbf{v}^h\right\rangle + \frac{u_0^2(\mathbf{x},\mathbf{p})}{4}|\mathbf{v}^h|^2$,

(2) $\rho(c,b)(\mathbf{v}) = \frac{1}{|\mathbf{p}^h|}\left\langle\mathbf{Rm}(J\mathbf{p}^h, \mathbf{p}^h)\mathbf{p}^h, \mathbf{v}^h\right\rangle$,

(3) $\rho(b,b) = \frac{1}{|\mathbf{p}^h|^2}\left\langle\mathbf{Rm}(J\mathbf{p}^h, \mathbf{p}^h)\mathbf{p}^h, J\mathbf{p}^h\right\rangle + u_0^2(\mathbf{x},\mathbf{p})$,

(4) $\mathfrak{R}(c,a) = 0$,

(5) $\mathfrak{R}(a,a) = 0$.

In general, the expression for $\rho(a,a)$ is very complicated and it is difficult to check whether it is nonnegative or not. Instead, we only mention the following partial result (see Appendix II for the proof).

Theorem 5.7. *The invariant $\rho(a,a)$ is a cubic polynomial on u_0. In addition, if we assume that $J^2 = -I$ (i.e. $(\langle\cdot,\cdot\rangle, J)$ is an almost Kähler structure on $\widetilde{M}$), then the coefficient of the u_0^3-term in $\rho(a,a)$ vanishes and the u_0^2-term is*

$$\frac{4u_0^2(\mathbf{x},\mathbf{p})}{|\mathbf{p}^h|^2}|\nabla_{\mathbf{p}^h}J(J\mathbf{p}^h)|^2 + \frac{8u_0^2(\mathbf{x},\mathbf{p})}{|\mathbf{p}^h|^2}|\nabla_{J\mathbf{p}^h}J(\mathbf{p}^h)|^2 + \frac{12u_0^2(\mathbf{x},\mathbf{p})}{|\mathbf{p}^h|^2}\left\langle \nabla_{\mathbf{p}^h}J(J\mathbf{p}^h), \nabla_{J\mathbf{p}^h}J(\mathbf{p}^h)\right\rangle.$$

Remark 5.2. *Theorem 5.7 shows that one can use the method in this paper to study measure contraction property for almost Kähler manifolds.*

Now let us investigate the component $\mathfrak{R}(c,c)$ in order to apply Lemma 4.3. As a direct consequence of Theorem 5.4, we obtain

Theorem 5.8. *Let $\mathbf{Rc}$ be the Ricci curvature tensor of $(\widetilde{M}, \langle\cdot,\cdot\rangle)$. Then*

$$\boldsymbol{tr}(\mathfrak{R}(c,c)) = \mathbf{Rc}(\mathbf{p}^h,\mathbf{p}^h) - \frac{1}{|J\mathbf{p}^h|^2}\left\langle \mathbf{Rm}(J\mathbf{p}^h,\mathbf{p}^h)\mathbf{p}^h, J\mathbf{p}^h\right\rangle + \boldsymbol{tr}(\mathcal{T}_{(\mathbf{x},\mathbf{p})}),$$

where $\mathcal{T}_{(\mathbf{x},\mathbf{p})}$ is a $(1,1)$-tensor on $\widetilde{M}$ defined by

$$\begin{aligned}\mathcal{T}_{(\mathbf{x},\mathbf{p})} = &-\frac{1}{|J\mathbf{p}^h|^2}\left\langle \mathbf{v}^h, \nabla_{\mathbf{p}^h}J(\mathbf{p}^h)\right\rangle \nabla_{\mathbf{p}^h}J(\mathbf{p}^h)\\ &+ u_0(\mathbf{x},\mathbf{p})\nabla_{\mathbf{v}^h}J(\mathbf{p}^h) + \frac{u_0(\mathbf{x},\mathbf{p})}{|J\mathbf{p}^h|^2}\left\langle \mathbf{v}^h, \nabla_{\mathbf{p}^h}J(\mathbf{p}^h)\right\rangle J^2\mathbf{p}^h\\ &- \frac{1}{4}u_0^2(\mathbf{x},\mathbf{p})J^2\mathbf{v}^h - \frac{u_0^2(\mathbf{x},\mathbf{p})}{4|J\mathbf{p}^h|^2}\left\langle \mathbf{v}^h, J^2\mathbf{p}^h\right\rangle J^2\mathbf{p}^h.\end{aligned}$$

Corollary 5.6. *In the almost Kählerian case ($J^2 = -I$),*

$$\begin{aligned}\boldsymbol{tr}(\mathfrak{R}(c,c)) = &\,\mathbf{Rc}(\mathbf{p}^h,\mathbf{p}^h) - \frac{1}{|\mathbf{p}^h|^2}\left\langle \mathbf{Rm}(J\mathbf{p}^h,\mathbf{p}^h)\mathbf{p}^h, J\mathbf{p}^h\right\rangle\\ &+ \frac{n-1}{2}u_0^2(\mathbf{x},\mathbf{p}) - \frac{|\nabla_{\mathbf{p}^h}J(\mathbf{p}^h)|^2}{|\mathbf{p}^h|^2} + u_0(\mathbf{x},\mathbf{p})\boldsymbol{tr}\left\langle \nabla_{\mathbf{v}^h}J(\mathbf{p}^h), \mathbf{v}^h\right\rangle,\end{aligned}$$

where $\boldsymbol{tr}$ is the trace taken with respect to the Riemannian metric on the quotient $\widetilde{M}$.

Corollary 5.7. *Assume that J is parallel. Then*

$$\boldsymbol{tr}(\Re(c,c)) = \mathbf{Rc}(\mathbf{p}^h,\mathbf{p}^h) - \frac{1}{|J\mathbf{p}^h|^2}\left\langle \mathbf{Rm}(J\mathbf{p}^h,\mathbf{p}^h)\mathbf{p}^h, J\mathbf{p}^h\right\rangle$$
$$- \frac{u_0^2(\mathbf{x},\mathbf{p})}{4}\left(\boldsymbol{tr}(J^2) + 2\frac{|J^2\mathbf{p}^h|^2}{|J\mathbf{p}^h|^2}\right).$$

Corollary 5.8. *Assume that $J^2 = -I$ and $\nabla J = 0$ (i.e. the Kähler case),*

$$\boldsymbol{tr}(\Re(c,c)) = \mathbf{Rc}(\mathbf{p}^h,\mathbf{p}^h) - \frac{1}{|\mathbf{p}^h|^2}\left\langle \mathbf{Rm}(J\mathbf{p}^h,\mathbf{p}^h)\mathbf{p}^h, J\mathbf{p}^h\right\rangle + \frac{n-1}{2}u_0^2(\mathbf{x},\mathbf{p}).$$

6. Proof of the Main Results

In this section, we finish the proof of the main results. Let $E(t), F(t)$ be a canonical frame at a point $(\mathbf{x},\mathbf{p})$ in the cotangent bundle T^*M. Let $\mathcal{E}_z$ and $\mathcal{F}_z$ be two vector fields defined locally near $(\mathbf{x},\mathbf{p})$ and satisfy the following conditions

$$\mathcal{E}_z(e^{t\vec{\mathbf{H}}}(\mathbf{x},\mathbf{p})) = e_*^{t\vec{\mathbf{H}}}E_z(t),\ \mathcal{F}_z(e^{t\vec{\mathbf{H}}}(\mathbf{x},\mathbf{p})) = e_*^{t\vec{\mathbf{H}}}F_z(t), \tag{6.13}$$

where $z = a, b, c_1, ..., c_{2n-1}$.

We also let

$$\varsigma_z = \pi_*(\mathcal{F}_z(\mathbf{x},\mathbf{p})).$$

We start with the following lemma which was needed in the proof of Lemma 3.1.

Lemma 6.4. *The above frame satisfies the followings:*

1. $\varsigma_b, \varsigma_{c_1}, \varsigma_{c_2} ..., \varsigma_{c_{2n-1}}$ *is orthonormal with respect to the sub-Riemannian metric* $\langle\cdot,\cdot\rangle$,
2. $\mu(\varsigma_a, \varsigma_b, \varsigma_{c_1}, ..., \varsigma_{c_{2n-1}}) = |J\mathbf{p}^h|$.

Proof. A computation shows that for any vector field V contained in the vertical bundle **ver**, there holds

$$(\pi_0\circ\pi)_*([\vec{\mathbf{H}}, V]) = -V^h,$$

where we recall that $\pi_0 : M \to \widetilde{M}$ is the quotient map. On the other hand, from Theorem 3.3 and (6.13), we have

$$\mathcal{F}_b(\mathbf{x},\mathbf{p}) = \frac{d}{dt}e_*^{-t\vec{\mathbf{H}}}\mathcal{E}_b(e^{t\vec{\mathbf{H}}}(\mathbf{x},\mathbf{p}))\Big|_{t=0} = [\vec{\mathbf{H}}, \mathcal{E}_b](\mathbf{x},\mathbf{p}),$$

and

$$\mathcal{F}_{c_i}(\mathbf{x},\mathbf{p}) = \frac{d}{dt}e_*^{-t\vec{\mathbf{H}}}\mathcal{E}_{c_i}(e^{t\vec{h}}(\mathbf{x},\mathbf{p}))\Big|_{t=0} = [\vec{\mathbf{H}}, \mathcal{E}_{c_i}](\mathbf{x},\mathbf{p}),$$

where $i = 1, ..., 2n-1$.

It follows that

$$(\pi_0)_*(\varsigma_b(\mathbf{x},\mathbf{p})) = -(\mathcal{E}_b(\mathbf{x},\mathbf{p}))^h$$

and

$$(\pi_0)_*(\varsigma_{c_i}(\mathbf{x},\mathbf{p})) = -(\mathcal{E}_{c_i}(\mathbf{x},\mathbf{p}))^h,$$

where $i = 1, ..., 2n-1$.

Since $(\pi_0)_* : \mathcal{D} \to T\widetilde{M}$ is a Riemannian isometry, it suffices to show

$$(\mathcal{E}_b(\mathbf{x},\mathbf{p}))^h, (\mathcal{E}_{c_1}(\mathbf{x},\mathbf{p}))^h, ..., (\mathcal{E}_{c_{2n-1}}(\mathbf{x},\mathbf{p}))^h$$

is orthonormal. Indeed, it follows from Theorem 3.3 that

$$\begin{aligned}\langle(\mathcal{E}_{c_i}(\mathbf{x},\mathbf{p}))^h, (\mathcal{E}_{c_j}(\mathbf{x},\mathbf{p}))^h\rangle &= \omega(\mathcal{E}_{c_i}(\mathbf{x},\mathbf{p}), \mathcal{F}_{c_j}(\mathbf{x},\mathbf{p})) \\ &= \omega(E_{c_i}(0), F_{c_j}(0)) = \delta_{ij}.\end{aligned}$$

The orthonormal relations involving $(\mathcal{E}_b(\mathbf{x},\mathbf{p}))^h$ are proved in a similar way. This finishes the proof of the first assertion.

For the second assertion, it suffices to show that $\varsigma_a = -|J\mathbf{p}^h|V_0$. From Theorem 3.3, it follows

$$\langle(\mathcal{E}_b(\mathbf{x},\mathbf{p}))^h, (\pi_0)_*\varsigma_a\rangle = -\omega(\mathcal{E}_b(\mathbf{x},\mathbf{p}), \mathcal{F}_a(\mathbf{x},\mathbf{p})) = 0,$$
$$\langle(\mathcal{E}_{c_i}(\mathbf{x},\mathbf{p}))^h, (\pi_0)_*\varsigma_a\rangle = -\omega(\mathcal{E}_{c_i}(\mathbf{x},\mathbf{p}), \mathcal{F}_a(\mathbf{x},\mathbf{p})) = 0.$$

Hence $(\pi_0)_*\varsigma_a = 0$ and so we can assume ς_a is contained in the subspace spanned by V_0.

Let $\vec{u}_0$ be the Hamiltonian vector field of the Hamiltonian u_0. Since $(\pi_0)_*\vec{u}_0 = V_0$, we can assume

$$\mathcal{F}_a(\mathbf{x},\mathbf{p}) = f\vec{u}_0 + V,$$

where f is a function in the cotangent bundle T^*M and V is a vector in the vertical bundle.

By combining Theorem 3.3 and Proposition 5.1, we have

$$\begin{aligned}1 &= \omega(E_a(0), F_a(0)) = \omega(\mathcal{E}_a(\mathbf{x},\mathbf{p}), \mathcal{F}_a(\mathbf{x},\mathbf{p})) \\ &= \omega\left(\frac{\partial_{u_0}}{|J\mathbf{p}^h|}, f\vec{u}_0\right) = -\frac{f}{|J\mathbf{p}^h|}.\end{aligned}$$

Hence $f = -|J\mathbf{p}^h|$ as claimed. □

Next, we give the proof of Theorem 2.1. Note that Corollary 2.1 is an immediate consequence of Theorem 2.1 and Corollary 5.8. Corollary 2.2 and 2.3 are also consequences of the proof of Theorem 2.1 and equality case of Lemma 4.2.

Proof of Theorem 2.1. If $E(t), F(t)$ is a canonical frame at the point $(\mathbf{x}, d\mathfrak{f}_{\mathbf{x}})$ in the cotangent bundle T^*M, then

$$t \mapsto (de^{\tau\vec{\mathbf{H}}}(E(t+\tau)), de^{\tau\vec{\mathbf{H}}}(F(t+\tau)))$$

is a canonical frame at the point $e^{t\vec{\mathbf{H}}}(\mathbf{x}, d\mathfrak{f}_{\mathbf{x}})$. It follows from this that $R(t)$ is the matrix representation of the operator $\mathfrak{R}_{e^{t\vec{\mathbf{H}}}(\mathbf{x},d\mathfrak{f})}$ with respect to the frame $de^{\tau\vec{\mathbf{H}}}(E(\tau))$.

Since V_0 is a symmetry, u_0 is constant along the flow $e^{t\vec{\mathbf{H}}}$ (see for instance [18]). Therefore, by the assumptions and Lemma 5.7, $R_{bb}(t)$ and $\mathbf{tr}(R_{cc}(t))$ are bounded below by $k_b|\nabla_{\mathbf{hor}}\mathfrak{f}|^2 + \frac{u_0^2(\mathbf{x},d\mathfrak{f})}{\lambda_2^2}$ and $k_c(2n-2)|\nabla_{\mathbf{hor}}\mathfrak{f}|^2 - \frac{u_0^2(\mathbf{x},d\mathfrak{f})}{4}(\lambda_3 + 2\lambda_1^2)$, respectively. Therefore, the assumptions of Lemma 4.3 are satisfied. By combining this with Lemma 3.1, the result follows. □

7. Proof of Bonnet-Myer's type Theorem

In this section, we prove Theorem 2.2.

Proof of Theorem 2.2. Let $(\mathbf{x}, \mathbf{p})$ be the covector in T^*M such that the corresponding sub-Riemannian geodesic $\gamma(t) = \pi(e^{t\vec{\mathbf{H}}}(\mathbf{x}, \mathbf{p}))$ with $0 \le t \le 1$ is minimizing and the length is equal to the diameter of M with respect to the sub-Riemannian metric. It follows that (see [3]) $de^{t\vec{\mathbf{H}}}(\mathbf{ver}_{(\mathbf{x},\mathbf{p})})$ intersects $\mathbf{ver}_{e^{t\vec{\mathbf{H}}}(\mathbf{x},\mathbf{p})}$ transversely for all t in $(0,1)$. Let $E(t), F(t)$ be a canonical frame at the point $(\mathbf{x}, \mathbf{p})$. Let $A(t)$ and $B(t)$ be matrices defined by

$$E(0) = A(t)E(t) + B(t)F(t).$$

It follows from the above discussion that $B(t)$ is invertible for all t in $(0,1)$. Clearly, we also have $A(0) = I$ and $B(0) = 0$. Therefore, if $S(t) = B(t)^{-1}A(t)$, then we have

$$\dot{S}(t) - S(t)C_2S(t) + C_1^T S(t) + S(t)C_1 - R(t) = 0$$

with $\lim_{t\to 0} S(t)^{-1} = 0$.

Assume that $k_1 > 0$. Then, by arguing as in Lemma 4.3, we have

$$\mathbf{tr}(\tilde{C}_2 S_1(t)) \ge -\frac{\sqrt{\mathbf{k_1}}(\sin(t\sqrt{\mathbf{k_1}}) - t\sqrt{\mathbf{k_1}}\cos(t\sqrt{\mathbf{k_1}}))}{2 - 2\cos(t\sqrt{\mathbf{k_1}}) - t\sqrt{\mathbf{k_1}}\sin(t\sqrt{\mathbf{k_1}})} \tag{7.14}$$

where

$$\begin{aligned}
f(y) &= d(\mathbf{x}, y),\\
\mathbf{k_1}(y) &= f(y)^2\left(k_1 + \frac{(V_0 f)^2(y)}{\lambda_2^2}\right) \geq k_1 f(y)^2,\\
S(t) &= \begin{pmatrix} S_1(t) & S_2(t) & S_3(t)\\ S_2(t)^T & S_4(t) & S_5(t)\\ S_3(t)^T & S_5(t)^T & S_6(t)\end{pmatrix},\\
\tilde{C}_2 &= \begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}.
\end{aligned}$$

Similarly, if $k_2 > 0$, then

$$\mathbf{tr} S_4(t) \geq -\sqrt{\mathbf{k_2}}(2n-2)\cot\left(t\sqrt{\mathbf{k_2}}\right), \tag{7.15}$$

where

$$\mathbf{k_2}(y) = f(y)^2\left(k_2 - \frac{(V_0 f)^2(y)}{4(2n-2)}\left(\lambda_3 + 2\lambda_1^2\right)\right) \geq k_2 f(y)^2.$$

Therefore, if $\frac{2\pi}{\sqrt{k_1}d(x,y)} < 1$ or $\frac{\pi}{\sqrt{k_2}d(x,y)} < 1$, we obtain a contradiction since the right hand sides of (7.14) and (7.15) go to ∞ for some time t in $(0,1)$ in this case. □

8. Appendix I: Proofs of Theorems 5.4-5.6

First, we introduce another version of Jacobi curves $\bar{\mathfrak{J}}(\cdot)$ and the curvature $\bar{\mathfrak{R}}$, called reduced Jacobi curves and reduced curvature, respectively. Then we show that the curvature $\mathfrak{R}$ can be recovered from the curvature $\bar{\mathfrak{R}}$ (see Theorem 8.10 below), which make Theorems 5.4-5.6 to be the consequences of the results on the curvature $\bar{\mathfrak{R}}$ in [15].

Recall that the Hamiltonian $\mathbf{H}$ is constant along the flow $e^{t\vec{\mathbf{H}}}$, so we can define another curve $\bar{\mathfrak{J}}_{(\mathbf{x},\mathbf{p})}$, called reduced Jacobi curve at $(\mathbf{x},\mathbf{p})$, by

$$t \longmapsto \bar{\mathfrak{J}}_{(\mathbf{x},\mathbf{p})}(t) := \left(\mathfrak{J}_{(\mathbf{x},\mathbf{p})}(t) \cap \vec{\mathbf{H}}^{\angle}_{(\mathbf{x},\mathbf{p})}\right)/\mathbb{R}\vec{\mathbf{H}}_{(\mathbf{x},\mathbf{p})}. \tag{8.16}$$

Here $\vec{\mathbf{H}}^{\angle}$ denotes the skew orthogonal complement of $\vec{\mathbf{H}}$ with respect to the symplectic form ω.

The symplectic form ω descends to a symplectic form on $\vec{\mathbf{H}}^{\angle}_{(\mathbf{x},\mathbf{p})}/\mathbb{R}\vec{\mathbf{H}}_{(\mathbf{x},\mathbf{p})}$ and the reduced Jacobi curve $\bar{\mathfrak{J}}_{(\mathbf{x},\mathbf{p})}$ is a curve in the Lagrange Grassmannian of $\vec{\mathbf{H}}^{\angle}_{(\mathbf{x},\mathbf{p})}/\mathbb{R}\vec{\mathbf{H}}_{(\mathbf{x},\mathbf{p})}$. Conversely, we can recover the non-reduced

Jacobi curve from the reduced one. Indeed, the vertical space $\mathbf{ver}_{(\mathbf{x},\mathbf{p})}$ splits into the following direct sum

$$\mathbf{ver}_{(\mathbf{x},\mathbf{p})} = \left(\mathbf{ver}_{(\mathbf{x},\mathbf{p})} \cap \vec{\mathbf{H}}^{\angle}_{(\mathbf{x},\mathbf{p})}\right) \oplus \mathbb{R}\mathfrak{E}_{(\mathbf{x},\mathbf{p})}.$$

Here $\mathfrak{E}$ is the Euler field defined in local coordinates by $\mathfrak{E} = \sum_i p_i \partial_{p_i}$. A computation shows that $e^{t\vec{\mathbf{H}}}_* \mathfrak{E} = \mathfrak{E} - t\vec{\mathbf{H}}$. It follows that

$$\mathfrak{J}_{(\mathbf{x},\mathbf{p})}(t) = \left(\mathfrak{J}_{(\mathbf{x},\mathbf{p})}(t) \cap \vec{\mathbf{H}}^{\angle}_{(\mathbf{x},\mathbf{p})}\right) \oplus \mathbb{R}\left(\mathfrak{E} - t\vec{\mathbf{H}}\right). \tag{8.17}$$

From the right hand side of (8.17), it is clear that the curve $\mathfrak{J}_{(\mathbf{x},\mathbf{p})}$ is completely determined by the reduced Jacobi curve $\bar{\mathfrak{J}}_{(\mathbf{x},\mathbf{p})}$ as claimed.

In a similar way, we have the following descriptions of a normal moving frames $\{\bar{E}(t), \bar{F}(t)\}$ of the reduced Jacobi cuves $\bar{\mathfrak{J}}_{\mathbf{x},\mathbf{p}}$ also as a particular case of the results in [13, 14].

Theorem 8.9. *Assume that the distribution $\mathcal{D}$ is contact. Then there exists a one-parameter family of bases*

$$\bar{E}(t) = (\bar{E}_a(t), \bar{E}_b(t), \bar{E}_{c_1}(t), ..., \bar{E}_{c_{2n-2}}(t))^T$$
$$\bar{F}(t) = (\bar{F}_a(t), \bar{F}_b(t), \bar{F}_{c_1}(t), ..., \bar{F}_{c_{2n-2}}(t))^T$$

of the symplectic vector space

$$\vec{\mathbf{H}}^{\angle}_{(\mathbf{x},\mathbf{p})}/\mathbb{R}\vec{\mathbf{H}}_{(\mathbf{x},\mathbf{p})}$$

such that the followings hold for any t:

1. $\bar{\mathfrak{J}}_{(\mathbf{x},\mathbf{p})}(t) = \mathrm{span}\{\bar{E}_a(t), \bar{E}_b(t), \bar{E}_{c_1}(t), ..., \bar{E}_{c_{2n-2}}(t)\}$,
2. $\mathrm{span}\{\bar{F}_a(t), \bar{F}_b(t), \bar{F}_{c_1}(t), ..., \bar{F}_{c_{2n-2}}(t)\}$ *is a family of Lagrangian subspaces,*
3. $\omega(\bar{E}_a(t), \bar{F}_a(t)) = \omega(\bar{E}_b(t), \bar{F}_b(t)) = 1$,
4. $\omega(\bar{E}_{c_i}(t), \bar{F}_{c_j}(t)) = \delta_{ij}$,
5. $\dot{\bar{E}}(t) = \bar{C}_1\bar{E}(t) + \bar{C}_2\bar{F}(t)$,
6. $\dot{\bar{F}}(t) = -\bar{R}(t)\bar{E}(t) - \bar{C}_1^T\bar{F}(t)$,

where $\bar{R}(t)$ is a symmetric matrix, $\bar{C}_1$ and $\bar{C}_2$ are $2n \times 2n$ matrices defined by

1. $\tilde{C}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *is a 2×2 matrix,*
2. $\tilde{C}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *is a 2×2 matrix,*
3. $\bar{C}_1 = \begin{pmatrix} \tilde{C}_1 & O \\ O & O \end{pmatrix}$
4. $\bar{C}_2 = \begin{pmatrix} \tilde{C}_2 & O \\ O & I \end{pmatrix}$.

Moreover, a moving frame

$$\tilde{E}_a(t), \tilde{E}_b(t), \tilde{E}_{c_1}(t), ..., \tilde{E}_{c_{2n-1}}(t), \tilde{F}_a(t), \tilde{F}_b(t), \tilde{F}_{c_1}(t), ..., \tilde{F}_{c_{2n-1}}(t)$$

satisfies conditions (1)-(5) above if and only if

$$(\tilde{E}_a(t), \tilde{E}_b(t), \tilde{F}_a(t), \tilde{F}_b(t)) = \pm(\bar{E}_a(t), \bar{E}_b(t), \bar{F}_a(t), \bar{F}_b(t)) \tag{8.18}$$

and there exists a constant orthogonal matrix $\bar{U}$ *of size* $(2n-2)\times(2n-2)$ *such that for all* t

$$\tilde{E}_{c_i}(t) = \sum_{j=1}^{2n-2} \bar{U}_{ij}\bar{E}_{c_j}(t) \text{ and } \tilde{F}_{c_i}(t) = \sum_{j=1}^{2n-2} \bar{U}_{ij}\bar{F}_{c_j}(t). \tag{8.19}$$

Similar to the case of a non-reduced Jacobi curve, the linear map

$$\bar{\mathfrak{R}} : (\mathbf{ver}_{(\mathbf{x},\mathbf{p})} \cap \vec{\mathbf{H}}^{\angle}_{(\mathbf{x},\mathbf{p})})/\mathbb{R}\vec{\mathbf{H}}(\mathbf{x},\mathbf{p}) \to (\mathbf{ver}_{(\mathbf{x},\mathbf{p})} \cap \vec{\mathbf{H}}^{\angle}_{(\mathbf{x},\mathbf{p})})/\mathbb{R}\vec{\mathbf{H}}(\mathbf{x},\mathbf{p})$$

having the matrix $\bar{R}(0)$ with respect to the canonical frame $(\bar{E}(0), \bar{F}(0))$ is called the reduced curvature map.

Next, we investigate how the non-reduced curvature can be recovered from the reduced curvature. Since $\vec{\mathbf{H}}$ is transversal to the vertical spaces, we can identify $\bar{\mathfrak{J}}_{(\mathbf{x},\mathbf{p})}(0)$ with its representative in $\mathfrak{J}_{(\mathbf{x},\mathbf{p})}(0) \cap \vec{\mathbf{H}}^{\angle}_{(\mathbf{x},\mathbf{p})}$. Therefore, we can consider $\bar{\mathfrak{R}}$ as a linear map on $\mathbf{ver}_{(\mathbf{x},\mathbf{p})} \cap \vec{\mathbf{H}}^{\angle}_{(\mathbf{x},\mathbf{p})}$.

Theorem 8.10. *The curvature* $\mathfrak{R}$ *can be recovered from the operator* $\bar{\mathfrak{R}}$ *as follows:*

(1) $\mathfrak{R}|_{\mathbf{ver}_{(\mathbf{x},\mathbf{p})}\cap\vec{\mathbf{H}}^{\angle}_{(\mathbf{x},\mathbf{p})}} = \bar{\mathfrak{R}}$;
(2) $\mathfrak{R}(\mathfrak{E}) = 0$.

Proof. Let $\{\bar{E}(t), \bar{F}(t)\}$ be a normal moving frame (in $\vec{\mathbf{H}}^{\angle}_{(\mathbf{x},\mathbf{p})}/\mathbb{R}\vec{\mathbf{H}}_{(\mathbf{x},\mathbf{p})}$) for the curve $\bar{\mathfrak{J}}$. Under the aforementioned identification between the spaces $\bar{\mathfrak{J}}_{(\mathbf{x},\mathbf{p})}$ and $\mathfrak{J}_{(\mathbf{x},\mathbf{p})} \cap \vec{\mathbf{H}}^{\angle}_{(\mathbf{x},\mathbf{p})}$, $\bar{E}(t)$ corresponds to a basis of $\mathfrak{J}_{(\mathbf{x},\mathbf{p})} \cap \vec{\mathbf{H}}^{\angle}_{(\mathbf{x},\mathbf{p})}$, which will be denoted by the same symbol.

As before, write

$$\begin{aligned} \bar{E}(t) &= (\bar{E}_a(t), \bar{E}_b(t), \bar{E}_{c_1}(t), ..., \bar{E}_{c_{2n-2}}(t))^T, \\ \bar{E}_c(t) &= (\bar{E}_{c_1}(t), ..., \bar{E}_{c_{2n-2}}(t)). \end{aligned} \tag{8.20}$$

Then set

$$\begin{aligned} \bar{F}_b(t) &:= \dot{\bar{E}}_b(t) \\ \bar{F}_c(t) &:= \dot{\bar{E}}_c(t) \\ \bar{F}_a(t) &:= -\dot{\bar{F}}_b(t) - \bar{E}_b(t)\bar{R}_t(b,b) - \bar{E}_c(t)\bar{R}_t(b,c), \end{aligned}$$

where $\bar{R}_t(\cdot,\cdot)$ are as in Theorem 8.9.

Further, let $\epsilon(t) = (e^{-t\vec{\mathbf{H}}})_*\mathfrak{E}(e^{t\vec{\mathbf{H}}}(\mathbf{x},\mathbf{p}))$, then, by a computation, we have $\upsilon_0(t) := \dot{\epsilon}(t) = \vec{\mathbf{H}}(\mathbf{x},\mathbf{p})$ and so $\dot{\upsilon}_0(t) = 0$. Next, we let $\widehat{E}_c(t) := \{E_c(t), \varepsilon_0(t)\}$, $\widehat{F}_c(t) = \{F_c(t), \upsilon_0(t)\}$. We will show that the tuple

$$\{\bar{E}_a(t), \bar{E}_b(t), \widehat{E}_c(t), \bar{F}_a(t), \bar{F}_b(t), \widehat{F}_c(t)\}. \tag{8.21}$$

constitute a normal moving frame for non-reduced Jacobi curve $\mathfrak{J}_{(\mathbf{x},\mathbf{p})}$.

Let us first show that the frame (8.21) is a symplectic moving frame. By construction, it is sufficient to check that $\varepsilon_0(t)$ is skew orthogonal to the vectors $\bar{F}_a(t)$, $\bar{F}_b(t)$, and the vectors from the tuple $\bar{F}_c(t)$. Indeed, since $\mathfrak{J}_{(\mathbf{x},\mathbf{p})}$ is Lagrangian, one has $\omega(\varepsilon_0(t), \bar{E}_b(t)\big)= 0$. By differentiating this identity , we get

$$\omega(\dot{\varepsilon}_0(t), \bar{E}_b(t)\big)+\omega(\varepsilon_0(t), \bar{F}_b(t)\big)= 0. \tag{8.22}$$

Since $\dot{\varepsilon}_0(t) = \vec{\mathbf{H}}(\mathbf{x},\mathbf{p})$ and $\bar{E}_b(t) \in \vec{\mathbf{H}}(\mathbf{x},\mathbf{p})^{\angle}$, the first term of (8.22) is equal to zero, which gives

$$\omega(\varepsilon_0(t), \bar{F}_b(t)\big)= 0. \tag{8.23}$$

The identity $\omega\big(\varepsilon_0(t), \bar{F}_c(t)\big) = 0$ is proved in completely the same way. For the proof of $\omega\big(\varepsilon_0(t), \bar{F}_a(t)\big) = 0$, it is enough to prove that $\omega\big(\varepsilon_0(t), \dot{\bar{F}}_b(t)\big) = 0$. The latter follows by the same scheme by differentiating (8.23).

Furthermore, the moving symplectic frame satisfies the following structure equations

$$\begin{cases} \dot{\bar{E}}_a(t) = \bar{E}_b(t) \\ \dot{\bar{E}}_b(t) = \bar{F}_b(t) \\ \dot{\bar{E}}_c(t) = \bar{F}_c(t) \\ \dot{\varepsilon}_0(t) = \upsilon_0(t) \\ \dot{\bar{F}}_a(t) = -\bar{E}_a(t)\bar{R}_t(a,a) - \bar{E}_c(t)\bar{R}_t(a,c) \\ \dot{\bar{F}}_b(t) = -\bar{E}_b(t)\bar{R}_t(b,b) - \bar{E}_c(t)\bar{R}_t(b,c) - \bar{F}_a(t) \\ \dot{\bar{F}}_c(t) = -\bar{E}_a(t)\bar{R}_t(c,a) - \bar{E}_b(t)\bar{R}_t(c,b) - \bar{E}_c(t)\bar{R}_t(c,c) \\ \dot{\upsilon}_0(t) = 0 \end{cases} \tag{8.24}$$

This yields that the moving frame (8.21) is normal for the curve $\mathfrak{J}_{(\mathbf{x},\mathbf{p})}$ (see Theorem 3.3). This and the form of the structure equation (8.24) implies the statement of of the present theorem due to the uniqueness part of Theorem 3.3. □

Now the results in Theorems 5.4-5.6 are consequences of the calculations on the reduced curvature $\mathfrak{R}$ in Section 5 of [15].

9. Appendix II: Proof of Theorem 5.7

Since the tensor J is defined using a closed 2-form $d\alpha$, we have

Proposition 9.2. *For any vector fields X, Y, Z on $\widetilde{M}$,*

(1) $\langle X, \nabla_Z J(Y)\rangle + \langle Y, \nabla_X J(Z)\rangle + \langle Z, \nabla_Y J(X)\rangle = 0$,
(2) $\langle X, \nabla_Z J(Y)\rangle + \langle Y, \nabla_Z J(X)\rangle = 0$.

The following two propositions are consequences of definition of $\mathcal{A}$ (c.f. Theorem 5.4) and Proposition 4.2 in [15] .

Proposition 9.3. *Let* $\mathbf{v}$ *be a vector in* $\mathbf{ver}_c$. *Then*

$$\mathcal{A}^{(1)}(\mathbf{v}) = -\mathcal{A}(\mathbf{v})\mathcal{A}\left(\frac{(J\mathbf{p}^h)^v}{|J\mathbf{p}^h|}\right) + \frac{1}{|J\mathbf{p}^h|}\left\langle \mathbf{v}^h, 2\nabla^2 J(\mathbf{p}^h, \mathbf{p}^h, \mathbf{p}^h)\right\rangle$$
$$+ \frac{1}{|J\mathbf{p}^h|}\left\langle \mathbf{v}^h, -3u_0\nabla_{\mathbf{p}^h} J(J\mathbf{p}^h) - 2u_0\nabla_{J\mathbf{p}^h} J(\mathbf{p}^h) + \frac{1}{2}u_0^2 J^3\mathbf{p}^h\right\rangle,$$

Proposition 9.4. *Let* $\mathbf{v}$ *be a vector in* $\mathbf{ver}_c$. *Then*

$$\mathcal{A}^{(2)}(\mathbf{v}) = -\mathcal{A}^{(1)}(\mathbf{v})\mathcal{A}\left(\frac{(J\mathbf{p}^h)^v}{|J\mathbf{p}^h|}\right) - \mathcal{A}(\mathbf{v})\mathcal{A}^{(1)}\left(\frac{(J\mathbf{p}^h)^v}{|J\mathbf{p}^h|}\right)$$
$$+ \vec{\mathbf{H}}\left(\frac{1}{|J\mathbf{p}^h|}\right)\left\langle \mathbf{v}^h, 2\nabla^2 J(\mathbf{p}^h, \mathbf{p}^h, \mathbf{p}^h)\right\rangle$$
$$- \frac{1}{|J\mathbf{p}^h|^2}\left\langle J\mathbf{p}^h, \nabla^2 J(\mathbf{p}^h, \mathbf{p}^h, \mathbf{p}^h)\right\rangle \mathcal{A}(\mathbf{v})$$
$$+ \frac{1}{|J\mathbf{p}^h|}\left\langle \mathbf{v}^h, 2\nabla^3 J(\mathbf{p}^h, \mathbf{p}^h, \mathbf{p}^h, \mathbf{p}^h)\right\rangle$$
$$- \frac{u_0}{|J\mathbf{p}^h|}\left\langle \mathbf{v}^h, 2\nabla^2 J(J\mathbf{p}^h, \mathbf{p}^h, \mathbf{p}^h) + 2\nabla^2 J(\mathbf{p}^h, J\mathbf{p}^h, \mathbf{p}^h)\right.$$
$$\left. + 2\nabla^2 J(\mathbf{p}^h, \mathbf{p}^h, J\mathbf{p}^h) - J\nabla^2 J(\mathbf{p}^h, \mathbf{p}^h, \mathbf{p}^h)\right\rangle$$
$$+ \vec{\mathbf{H}}\left(\frac{1}{|J\mathbf{p}^h|}\right)\left\langle \mathbf{v}^h, -3u_0\nabla_{\mathbf{p}^h} J(J\mathbf{p}^h) - 2u_0\nabla_{J\mathbf{p}^h} J(\mathbf{p}^h) + \frac{1}{2}u_0^2 J^3\mathbf{p}^h\right\rangle$$
$$- \frac{\mathcal{A}(\mathbf{v})}{2|J\mathbf{p}^h|^2}\left\langle J\mathbf{p}^h, -3u_0\nabla_{\mathbf{p}^h} J(J\mathbf{p}^h) - 2u_0\nabla_{J\mathbf{p}^h} J(\mathbf{p}^h) + \frac{1}{2}u_0^2 J^3\mathbf{p}^h\right\rangle$$
$$+ \frac{1}{|J\mathbf{p}^h|}\left\langle \mathbf{v}^h, -3u_0\nabla^2 J(J\mathbf{p}^h, \mathbf{p}^h, \mathbf{p}^h) - 3u_0\nabla_{\mathbf{p}^h} J(\nabla_{\mathbf{p}^h} J(\mathbf{p}^h))\right\rangle$$
$$+ \frac{1}{|J\mathbf{p}^h|}\left\langle \mathbf{v}^h, -2u_0\nabla^2 J(\mathbf{p}^h, J\mathbf{p}^h, \mathbf{p}^h) - 2u_0\nabla_{\nabla_{\mathbf{p}^h} J(\mathbf{p}^h)} J(\mathbf{p}^h)\right\rangle$$

$$+\frac{1}{2|J\mathbf{p}^h|}\left\langle \mathbf{v}^h, u_0^2\nabla_{\mathbf{p}^h}(J^2\mathbf{p}^h)+u_0^2J\nabla_{\mathbf{p}^h}(J\mathbf{p}^h)+u_0^2J^2\nabla_{\mathbf{p}^h}(\mathbf{p}^h)\right\rangle$$
$$+\frac{1}{|J\mathbf{p}^h|}\Big\langle \mathbf{v}^h, 3u_0^2\nabla_{J\mathbf{p}^h}J(J\mathbf{p}^h)+3u_0^2\nabla_{\mathbf{p}^h}J(J^2\mathbf{p}^h)+2u_0^2\nabla_{J^2\mathbf{p}^h}J(\mathbf{p}^h)$$
$$+2u_0^2\nabla_{J\mathbf{p}^h}J(J\mathbf{p}^h)-\frac{1}{2}u_0^3J^4\mathbf{p}^h\Big\rangle$$
$$+\frac{1}{2|J\mathbf{p}^h|}\left\langle \mathbf{v}^h, -3u_0^2J\nabla_{\mathbf{p}^h}J(J\mathbf{p}^h)-2u_0J\nabla_{J\mathbf{p}^h}J(\mathbf{p}^h)+\frac{1}{2}u_0^3J^4\mathbf{p}^h\right\rangle$$

From now on, we assume $J^2=-I$. Then in addition to Proposition 9.2, we also have

Proposition 9.5. *For any vector fields X, Y on $\widetilde{M}$,*

(1) $\nabla_Y J(JX)+J\nabla_Y J(X)=0$,

(2) $\langle JX, \nabla_X J(X)\rangle = 0$.

Proof. The first item is from taking covariant derivative of J^2. For the second item, as Riemannian metric is compatible, then

$$\begin{aligned}\langle JX,\nabla_X J(X)\rangle &= \langle JX, \nabla_X JX - J\nabla_X X\rangle\\ &= \langle JX,\nabla_X JX\rangle - \langle X,\nabla_X X\rangle\\ &= \frac{1}{2}X(|JX|^2)-\frac{1}{2}X(|X|^2)=0.\end{aligned}$$

□

Next, we show that the u_0^3-term in $\rho(a,a)$ vanishes. From Proposition 9.3 and Proposition 9.5 it follows that

$$\left\langle J^2\mathbf{p}^h, \nabla_{\mathbf{p}^h}J(J\mathbf{p}^h)+\nabla_{J\mathbf{p}^h}J(\mathbf{p}^h)+J\nabla J_{\mathbf{p}^h}(\mathbf{p}^h)\right\rangle = 0.$$

and

$$\left\langle J^2\mathbf{p}^h, \nabla_{J^2\mathbf{p}^h}J(\mathbf{p}^h)\right\rangle + \left\langle J^2\mathbf{p}^h, \nabla_{\mathbf{p}^h}J(J^2\mathbf{p}^h)\right\rangle + \left\langle J^4\mathbf{p}^h, \nabla_{\mathbf{p}^h}J(\mathbf{p}^h)\right\rangle = 0$$

With the above two identities, together again that $J^2=-I$, the claim follows.

Finally, we analyze the u_0^2-term of $\rho(a,a)$. Note that in the present case, the vector $\mathfrak{V}_1$ contained in $\mathbf{ver}_c$ satisfies

$$\mathfrak{V}_1^h = -\frac{2}{|\mathbf{p}^h|}\nabla_{\mathbf{p}^h}J(\mathbf{p}^h).$$

Since $|J\mathbf{p}^h|=|\mathbf{p}^h|=\sqrt{2\mathbf{H}}$,

$$\rho(a,a)=\vec{\mathbf{H}}\left(\rho(c,b)(\mathfrak{V}_1)\right)+\rho_{(c,a)}(\mathfrak{V}_1). \tag{9.25}$$

From [15, Proposition 4.2], Propositions 9.3, and 9.5, one can get (after some calculations) that the u_0^2-term in $\rho(c,a)\mathbf{v}$ is

$$6u_0^2\left\langle \mathbf{v}^h, -\nabla_{\mathbf{p}^h} J(\mathbf{p}^h) + \nabla_{J\mathbf{p}^h} J(J\mathbf{p}^h)\right\rangle.$$

Furthermore, the only u_0-term of $\rho_{(c,b)}(\nabla_{\mathbf{p}^h} J(\mathbf{p}^h))$ is

$$4u_0\left\langle \nabla_{\mathbf{p}^h} J(\mathbf{p}^h), \nabla_{\mathbf{p}^h} J(J\mathbf{p}^h) + \nabla_{J\mathbf{p}^h} J(\mathbf{p}^h)\right\rangle.$$

By combining the above analysis with identity (9.25), we get the conclusion on u_0-term in $\rho(a,a)$ and thus complete the proof of Theorem 5.7.

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