disadvantage of the germ approach is simply that it adds an additional level of complication to an already highly abstract definition.

Another common approach to defining T_pM is to define an intrinsic equivalence relation on the set of smooth curves in M starting at p, which amounts to "having the same tangent vector," and to define a tangent vector as an equivalence class of curves. For example, one such equivalence relation is the following: If $\gamma_1 : J_1 \to M$ and $\gamma_2 : J_2 \to M$ are two smooth curves such that $\gamma_1(0) = \gamma_2(0) = p$, then we say $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Problem 3-8 shows that the set of equivalence classes is in one-to-one correspondence with T_pM. This definition has the advantage of being geometrically more intuitive, but it has the serious drawback that the existence of a vector space structure on T_pM is not at all obvious.

Yet another approach to defining the tangent space is based on the transformation rule (3.9) for the components of tangent vectors in coordinates. One defines a tangent vector at a point $p \in M$ to be a rule that assigns a vector $(X^1, \ldots, X^n) \in \mathbb{R}^n$ to each smooth coordinate chart containing p, with the property that the vectors assigned to overlapping charts transform according to (3.9). (This is, in fact, the oldest definition of all, and many physicists are still apt to define tangent vectors this way.)

It is a matter of individual taste which of the various characterizations of T_pM one chooses to take as the definition. The modern definition we have chosen, however abstract it may seem at first, has several advantages: It is relatively concrete (tangent vectors are actual derivations of $C^\infty(M)$, with no equivalence classes involved); it makes the vector space structure on T_pM obvious; and it leads to straightforward coordinate-independent definitions of many of the other geometric objects we will be studying.

Problems

3-1. Suppose M and N are smooth manifolds with M connected, and $F : M \to N$ is a smooth map such that $F_* : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$. Show that F is a constant map.

3-2. Let $M_1, \ldots, M_k$ be smooth manifolds, and let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection onto the jth factor. Show that for any choices of points $p_i \in M_i$, $i = 1, \ldots, k$, the map

$$\alpha : T_{(p_1,\ldots,p_k)}(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(X) = (\pi_{1*}X, \ldots, \pi_{k*}X)$$

is an isomorphism. [Remark: Using this isomorphism, we can routinely identify T_pM and T_qN, for example, as subspaces of $T_{(p,q)}(M \times N)$.]

3-3. If a nonempty smooth n-manifold is diffeomorphic to an m-manifold, prove that $n = m$.

3-4. Let $C \subset \mathbb{R}^2$ be the unit circle, and let $S \subset \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin:

$$S = \{(x, y) : \max(|x|, |y|) = 1\}.$$

Show that there is a homeomorphism $F: \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(C) = S$, but there is no *diffeomorphism* with the same property. [Hint: Consider what F does to the tangent vector to a suitable curve in C.]

3-5. Consider $\mathbb{S}^3$ as a subset of $\mathbb{C}^2$ under the usual identification of $\mathbb{C}^2$ with $\mathbb{R}^4$. For each $z = (z^1, z^2) \in \mathbb{S}^3$, define a curve $\gamma_z: \mathbb{R} \to \mathbb{S}^3$ by

$$\gamma_z(t) = \left(e^{it}z^1, e^{it}z^2\right).$$

(a) Compute the coordinate representation of $\gamma_z(t)$ in stereographic coordinates, and use this to show that γ_z is a smooth curve.

(b) Compute $\gamma_z'(t)$ in stereographic coordinates, and show that it is never zero.

3-6. Let G be a Lie group.

(a) Let $m: G \times G \to G$ denote the multiplication map. Identifying $T_{(e,e)}(G \times G)$ with $T_eG \oplus T_eG$ as in Problem 3-2, show that $m_*: T_eG \oplus T_eG \to T_eG$ is given by $m_*(X, Y) = X + Y$ [Hint: Compute $m_*(X, 0)$ and $m_*(0, Y)$ separately using (3.10).]

(b) Let $i: G \to G$ denote the inversion map. Show that $i_*: T_eG \to T_eG$ is given by $i_*X = -X$.

3-7. Let M be a smooth manifold. For any point $p \in M$, let C_p^∞ denote the algebra of germs of smooth real-valued functions at p, and let $\mathcal{D}_p$ denote the vector space of derivations of C_p^∞ at p. Show that T_pM is naturally isomorphic to $\mathcal{D}_p$.

3-8. Let M be a smooth manifold and $p \in M$. Let $\mathcal{C}_p$ denote the set of smooth curves $\gamma: J \to M$ such that $0 \in J$ and $\gamma(0) = p$. Define an equivalence relation on $\mathcal{C}_p$ by saying that $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p, and let $\mathcal{V}_p$ denote the set of equivalence classes. Show that the map $\Phi: \mathcal{V}_p \to T_pM$ defined by $\Phi[\gamma] = \gamma'(0)$ is well-defined and yields a one-to-one correspondence between $\mathcal{V}_p$ and T_pM.