4-3. Let M be a nonempty manifold of dimension $n \geq 1$. Show that $\mathcal{T}(M)$ is infinite-dimensional.

4-4. Show by finding a counterexample that Proposition 4.10 is false if we assume merely that F is smooth and bijective but not a diffeomorphism.

4-5. For each of the following vector fields on the plane, compute its coordinate representation in polar coordinates on the right half-plane $\{(x, y) : x > 0\}$.

(a) $V = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$.

(b) $W = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$.

(c) $X = (x^2 + y^2)\dfrac{\partial}{\partial x}$.

4-6. Show that there is a smooth vector field on $\mathbb{S}^2$ that vanishes at exactly one point. [Hint: Try using stereographic projection.]

4-7. Let M, N be smooth manifolds, and let $f\colon M \to N$ be a smooth map. Define $F\colon M \to M \times N$ by $F(x) = (x, f(x))$. Show that for every $V \in \mathcal{T}(M)$, there is a smooth vector field on $M \times N$ that is F-related to V.

4-8. Let $M_1, \ldots, M_k$ be smooth manifolds, and for each $i = 1, \ldots, k$, let $\pi_i\colon M_1 \times \cdots \times M_k \to M_i$ be the projection onto the ith factor. Show that for every $X \in \mathcal{T}(M_i)$, there is a smooth vector field on $M_1 \times \cdots \times M_k$ that is π_i-related to X.

4-9. Let $F\colon M \to N$ be a local diffeomorphism. Show that for every $Y \in \mathcal{T}(N)$, there is a unique smooth vector field on M that is F-related to Y.

4-10. Define a (rough) vector field V on $\mathbb{S}^3$ by letting $V_z = \gamma_z'(0)$, where γ_z is the curve of Problem 3-5. Show that V is smooth and nowhere vanishing.

4-11. For each of the following pairs of vector fields V, W defined on $\mathbb{R}^3$, compute the Lie bracket $[V, W]$.

(a) $V = y\dfrac{\partial}{\partial z} - 2xy^2\dfrac{\partial}{\partial y}$; $\quad W = \dfrac{\partial}{\partial y}$.

(b) $V = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$; $\quad W = y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y}$.

(c) $V = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$; $\quad W = x\dfrac{\partial}{\partial y} + y\dfrac{\partial}{\partial x}$.

4-12. Show that $\mathbb{R}^3$ with the cross product is a Lie algebra.