8-16. This problem gives a generalization of the regular level set theorem. Suppose $\Phi\colon M \to N$ is a smooth map and $S \subset N$ is an embedded submanifold. We say that Φ is *transverse to* S if for every $p \in \Phi^{-1}(S)$, the spaces $T_{\Phi(p)}S$ and $\Phi_* T_p M$ together span $T_{\Phi(p)}N$. If Φ is transverse to S, show that $\Phi^{-1}(S)$ is an embedded submanifold of M whose codimension is equal to $\dim N - \dim S$.

8-17. Let M be a smooth manifold. Two embedded submanifolds $S_1, S_2 \subset M$ are said to be *transverse* (or to *intersect transversely*) if for each $p \in S_1 \cap S_2$, the tangent spaces T_pS_1 and T_pS_2 together span T_pM. If S_1 and S_2 are transverse, show that $S_1 \cap S_2$ is an embedded submanifold of M of dimension $\dim S_1 + \dim S_2 - \dim M$. Give a counterexample when S_1 and S_2 are not transverse.

8-18. Let M be a smooth manifold, and let $C \subset M$ be an embedded submanifold that admits a global defining function $\Phi\colon M \to \mathbb{R}^k$. Let $f \in C^\infty(M)$, and suppose $p \in C$ is a point at which f attains its maximum or minimum value among points in C. Show that there are real numbers $\lambda_1, \dots, \lambda_k$ (called *Lagrange multipliers*) such that

$$df_p = \lambda_1 d\Phi^1|_p + \cdots + \lambda_k d\Phi^k|_p.$$

8-19. Let $\mathbb{H} = \mathbb{C} \times \mathbb{C}$ (considered as a real vector space), and define a bilinear product $\mathbb{H} \times \mathbb{H} \to \mathbb{H}$ by

$$(a,b)(c,d) = \left(ac - \bar{d}b, da + b\bar{c}\right), \qquad \text{for } a,b,c,d \in \mathbb{C}.$$

With this product, $\mathbb{H}$ is a 4-dimensional algebra over $\mathbb{R}$, called the algebra of *quaternions*. For any $p = (a,b) \in \mathbb{H}$, define $p^* = (\bar{a}, -b)$. It is useful to work with the basis $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$ for $\mathbb{H}$ defined by

$$\mathbf{1} = (1,0), \quad \mathbf{i} = (i,0), \quad \mathbf{j} = (0,1), \quad \mathbf{k} = (0,i).$$

It is straightforward to verify that this basis satisfies

$$\begin{aligned} \mathbf{1}q = q\mathbf{1} &= q \qquad \text{for all } q \in \mathbb{H}, \\ \mathbf{ij} &= -\mathbf{ji} = \mathbf{k}, \\ \mathbf{jk} &= -\mathbf{kj} = \mathbf{i}, \\ \mathbf{ki} &= -\mathbf{ik} = \mathbf{j}, \\ \mathbf{i}^2 = \mathbf{j}^2 &= \mathbf{k}^2 = -\mathbf{1}, \end{aligned}$$

and

$$\left(x^1\mathbf{1} + x^2\mathbf{i} + x^3\mathbf{j} + x^4\mathbf{k}\right)^* = x^1\mathbf{1} - x^2\mathbf{i} - x^3\mathbf{j} - x^4\mathbf{k}.$$

A quaternion p is said to be *real* if $p^* = p$, and *imaginary* if $p^* = -p$. Real quaternions can be identified with real numbers via the correspondence $x \leftrightarrow x\mathbf{1}$.

(a) Show that quaternionic multiplication is associative but not commutative.