neither empty nor all of S, then $L_p \cup S$ is a connected integral manifold properly containing L_p, which contradicts the maximality of L_p. Therefore, $L_p \cap U$ is precisely a countable union of slices, so the collection $\{L_p : p \in M\}$ is the desired foliation. □

Problems

19-1. Let M be a smooth n-manifold. An ideal $\mathcal{I} \subset \mathcal{A}^*(M)$ is called a *Pfaffian system* if for some $k \leq n$, $\mathcal{I}$ is locally generated by $n-k$ independent 1-forms, in the following sense: There exist open sets $\{U_\alpha\}_{\alpha \in A}$ covering M, and for each $\alpha \in A$ there exist $n-k$ independent smooth 1-forms $\omega_\alpha^1, \ldots, \omega_\alpha^{n-k}$ on U_α, such that an element $\eta \in \mathcal{A}^*(M)$ is in $\mathcal{I}$ if and only if for each $\alpha \in A$, the restriction of η to U_α is of the form

$$\eta|_{U_\alpha} = \sum_{i=1}^{n-k} \omega_\alpha^i \wedge \beta_\alpha^i$$

for some $\beta_\alpha^1, \ldots, \beta_\alpha^{n-k} \in \mathcal{A}^*(U_\alpha)$. Show that an ideal $\mathcal{I} \subset \mathcal{A}^*(M)$ is a Pfaffian system if and only if there is a smooth distribution D on M such that $\mathcal{I} = \mathcal{I}(D)$.

19-2. Let D be a smooth k-dimensional distribution on a smooth n-manifold M, and suppose $\omega^1, \ldots, \omega^{n-k}$ are smooth local defining forms for D on an open set $U \subset M$. Show that D is involutive on U if and only if the following identity holds for each $i = 1, \ldots, n-k$:

$$d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^{n-k} = 0.$$

19-3. Let $D \subset TM$ be a smooth distribution, and let $\mathcal{D}(M) \subset \mathcal{T}(M)$ denote the space of smooth global sections of D. Show that D is involutive if and only if $\mathcal{D}(M)$ is a Lie subalgebra of $\mathcal{T}(M)$.

19-4. If H is an integral manifold of an involutive distribution on N, and $\gamma : J \to N$ is a smooth curve whose image lies in H, show that $\gamma'(t)$ is in $T_{\gamma(t)}H \subset T_{\gamma(t)}N$ for all $t \in J$. [Remark: Compare this to the result of Problem 8-14.]

19-5. Let ω be a smooth 1-form on a smooth manifold M. A smooth positive function μ on some open subset $U \subset M$ is called an *integrating factor* for ω if $\mu\omega$ is exact on U.

(a) If ω is nowhere-vanishing, show that ω admits an integrating factor in a neighborhood of each point if and only if $d\omega \wedge \omega = 0$.

(b) If $\dim M = 2$, show that every nonvanishing smooth 1-form admits an integrating factor in a neighborhood of each point.