should be careful to think about which role σ is playing at each step of the argument.

In terms of any smooth local coordinates (x^i) for M and the corresponding standard coordinates (x^i, ξ_i) for T^*M, the map $\sigma\colon M \to T^*M$ has the coordinate representation

$$\sigma(x^1, \ldots, x^n) = (x^1, \ldots, x^n, \sigma_1(x), \ldots, \sigma_n(x)),$$

where $\sigma_i\, dx^i$ is the coordinate representation of σ as a 1-form. It follows immediately that σ is an immersion, and the fact that it is injective follows from $\pi \circ \sigma = \mathrm{Id}_M$.

To show that it is an embedding, it suffices by Proposition 7.4 to show that it is a proper map. This in turn follows from the fact that π is a left inverse for σ, by Lemma 2.16.

Because $\sigma(M)$ is n-dimensional, it is Lagrangian if and only if it is isotropic, which is the case if and only if $\sigma^*\omega = 0$. The pullback of the tautological form τ under σ is

$$\sigma^*\tau = \sigma^*(\xi_i\, dx^i) = \sigma_i\, dx^i = \sigma.$$

This can also be seen somewhat more invariantly from the computation

$$(\sigma^*\tau)_p(X) = \tau_{\sigma(p)}(\sigma_* X) = \sigma_p(\pi_* \sigma_* X) = \sigma_p(X),$$

which follows from the definition of τ and the fact that $\pi \circ \sigma = \mathrm{Id}_M$. Therefore,

$$\sigma^*\omega = -\sigma^* d\tau = -d(\sigma^*\tau) = -d\sigma.$$

It follows that σ is a Lagrangian embedding if and only if $d\sigma = 0$. □

Problems

12-1. Let $v_1, \ldots, v_n$ be any n vectors in $\mathbb{R}^n$, and let P be the n-dimensional parallelepiped spanned by them:

$$P = \{t_1 v_1 + \cdots + t_n v_n : 0 \le t_i \le 1\}.$$

Show that $\mathrm{Vol}(P) = |\det(v_1, \ldots, v_n)|$.

12-2. Let (e^1, e^2, e^3) be the standard dual basis for $(\mathbb{R}^3)^*$. Show that $e^1 \otimes e^2 \otimes e^3$ is not equal to a sum of an alternating tensor and a symmetric tensor.

12-3. Show that covectors $\omega^1, \ldots, \omega^k$ on a finite-dimensional vector space are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.

12-4. Show that two k-tuples $\{\omega^1, \ldots, \omega^k\}$ and $\{\eta^1, \ldots, \eta^k\}$ of independent covectors have the same span if and only if

$$\omega^1 \wedge \cdots \wedge \omega^k = c\,\eta^1 \wedge \cdots \wedge \eta^k$$

for some nonzero real number c.

12-5. A k-covector η on a finite-dimensional vector space V is said to be *decomposable* if it can be written

$$\eta = \omega^1 \wedge \cdots \wedge \omega^k,$$

where $\omega^1, \ldots, \omega^k$ are covectors. For what values of n is it true that every 2-covector on $\mathbb{R}^n$ is decomposable?

12-6. Define a 2-form Ω on $\mathbb{R}^3$ by

$$\Omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy.$$

(a) Compute Ω in spherical coordinates (ρ, φ, θ) defined by $(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)$.
(b) Compute $d\Omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.
(c) Compute the restriction $\Omega|_{S^2} = \iota^*\Omega$, using coordinates (φ, θ), on the open subset where these coordinates are defined.
(d) Show that $\Omega|_{S^2}$ is nowhere zero.

12-7. In each of the following problems, $g\colon M \to N$ is a smooth map between manifolds M and N, and ω is a smooth differential form on N. In each case, compute $g^*\omega$ and $d\omega$, and verify by direct computation that $g^*(d\omega) = d(g^*\omega)$.

(a) $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$(x, y) = g(s, t) = (st, e^t);$$
$$\omega = x\,dy.$$

(b) $g\colon \mathbb{R}^2 \to \mathbb{R}^3$ by

$$(x, y, z) = g(\theta, \varphi) = ((\cos\varphi + 2)\cos\theta, (\cos\varphi + 2)\sin\theta, \sin\varphi);$$
$$\omega = y\,dz \wedge dx.$$

(c) $g\colon \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\} \to \mathbb{R}^3 \setminus \{0\}$ by

$$(x, y, z) = g(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right);$$
$$\omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

12-8. Let V be a finite-dimensional real vector space. We have two ways to think about the tensor space $T^k(V)$: concretely, as the space of k-multilinear functionals on V; and abstractly, as the tensor product space $V^* \otimes \cdots \otimes V^*$. However, we have defined alternating and symmetric tensors only in terms of the concrete definition. This problem outlines an abstract approach to alternating tensors. (Symmetric tensors can be handled similarly.)