Controllability of invariant systems on Lie groups and homogeneous spaces (for beginners)*

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Contents

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1 Motivation

1.1 Bilinear systems

In the study of controllability of a bilinear control system

$$\dot{x} = Ax + uBx, \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}, \tag{1}$$

where A and B are constant $n \times n$ matrices, one naturally passes from the system (1) for vectors to the similar system for matrices:

$$\dot{X} = AX + uBX, \qquad X \quad n \times n \text{ matrix}, \quad u \in \mathbb{R}. \tag{2}$$

Such a passage is very natural: recall that in the study of the linear ODE $\dot{x} = Ax$, we pass to the matrix ODE $\dot{X} = AX$, here X is the Cauchy matrix for the linear ODE.

There is a clear and important relation between controllability properties of the bilinear system (1) and the matrix system (2):

“ system (2) is controllable $\quad\Rightarrow\quad$ system (1) is controllable ”

In the sequel we make this statement precise and remove the quotation marks. But this implication is clear: if we can control on matrices, the more so we can control on vectors.

One may think that matrix systems (2) are more complicated then the bilinear ones (1), but this is not the case: the matrix systems evolve on matrix groups (linear Lie groups), while the bilinear ones just on smooth submanifolds of $\mathbb{R}^n$ (homogeneous spaces of linear Lie groups). And the study of controllability for matrix systems is an easier problem since here the group structure provides powerful additional techniques. We will clarify all these questions in our course.

1.2 Rotations of a rigid body

Some important control systems in mechanics, physics, geometry etc. naturally evolve on groups.

Consider rotations of a rigid body in $\mathbb{R}^3$ around a fixed point (e.g. rotations of a space satellite around its center of mass). In order to describe motion of the body, choose a fixed orthonormal frame e_1, e_2, e_3 in the ambient space, and a moving orthonormal frame f_1, f_2, f_3 attached to the body. Then the orientation matrix

$$X \;:\; (e_1, e_2, e_3) \mapsto (f_1, f_2, f_3)$$

is a 3×3 orthogonal unimodular matrix. Moreover, it is easy to see that the matrix $\dot{X}X^{-1} = \Omega$ is skew-symmetric (the angular velocity of the body). So we obtain the equation of motion

$$\dot{X} = \Omega X.$$

If we suppose that we can control the matrix Ω, than the previous system is a matrix control system similar to (2). In the study of such systems, many questions arise, and one of the first ones is, what is the state space of such systems:

$$X \in ?$$

We will answer this question in the next section.

2 Lie groups and Lie algebras

2.1 Linear Lie groups

The most important class of Lie groups is formed by linear Lie groups, i.e., groups of linear transformations of $\mathbb{R}^n$.

Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping. In a basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, the operator X has a matrix $X = (x_{ij})$, $i, j = 1, \ldots, n$, which we identify with the operator itself. So we are going to consider groups of matrices.

Denote the linear space of all $n \times n$ matrices with real entries as

$$M(n, \mathbb{R}) = \{X = (x_{ij}) \mid x_{ij} \in \mathbb{R},\ i, j = 1, \ldots, n\}.$$

For short, we will usually denote this space by $M(n)$. The matrix entries x_{ij} provide coordinates on $M(n) = \mathbb{R}^{n^2}$.

Example 2.1 (General Linear Group). The *general linear group* consists of all $n \times n$ invertible matrices:

$$\mathrm{GL}(n, \mathbb{R}) = \mathrm{GL}(n) = \{X \in M(n) \mid \det X \neq 0\}.$$

The following properties of $\mathrm{GL}(n)$ are easily established.

(1) By continuity of determinant, $\det : M(n) \to \mathbb{R}$, the set $\mathrm{GL}(n)$ is an open domain, thus a smooth submanifold in the linear space $M(n)$.

(2) Further, $\mathrm{GL}(n)$ is a group with respect to matrix product. Indeed, if $X, Y \in \mathrm{GL}(n)$, then the product $XY \in \mathrm{GL}(n)$. Further, the identity matrix $\mathrm{Id} = (\delta_{ij})$ (where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$, the Kronecker symbol) is contained in $\mathrm{GL}(n)$. Finally, for a nonsingular matrix X, its inverse X^{-1} is nonsingular as well.

(3) Moreover, the group operations in $\mathrm{GL}(n)$ are smooth:

$$\begin{aligned} &(X, Y) \mapsto XY && (XY)_{ij} \text{ are polynomials in } X_{ij}, Y_{ij}, \\ &X \mapsto X^{-1} && (X^{-1})_{ij} \text{ are rational functions in } X_{ij}. \end{aligned}$$

Definition 2.1. A set G is called a *Lie group* if:

(1) G is a smooth manifold,

(2) G is a group, and

(3) the group operations in G are smooth.

In the previous example we showed that $\mathrm{GL}(n)$ is a Lie group.

Definition 2.2. A Lie group $G \subset M(n)$ is called a *linear Lie group*.

An convenient sufficient condition for a set of matrices to form a linear Lie group is given in the following general proposition.

Theorem 2.1. *If G is a closed subgroup of $\mathrm{GL}(n)$, then G is a linear Lie group.*

Proof. See e.g. [23]. □

In other words, in order to verify that a set of matrices $G \subset M(n)$ is a linear Lie group, it suffices to show that the following three conditions hold:

(1) $G \subset \mathrm{GL}(n)$,

(2) G is a group w.r.t. matrix product, and

(3) G is topologically closed in $\mathrm{GL}(n)$ (i.e., $G = \mathrm{GL}(n) \cap S$, where S is a closed subset in $M(n)$).

Now we consider several important examples of linear Lie groups in addition to the largest one, $\mathrm{GL}(n)$. In all these cases the hypotheses of Theorem 2.1 are easily verified.

Example 2.2 (Special Linear Group). The *special linear group* consists of $n \times n$ unimodular matrices:

$$\mathrm{SL}(n, \mathbb{R}) = \mathrm{SL}(n) = \{X \in M(n) \mid \det X = 1\}.$$

Such matrices correspond to linear operators preserving the standard volume in $\mathbb{R}^n$.

Example 2.3 (Orthogonal Group). The *orthogonal group* is formed by $n \times n$ orthogonal matrices:

$$\mathrm{O}(n) = \{X \in M(n) \mid XX^{\mathrm{T}} = \mathrm{Id}\},$$

where X^{T} denotes the transposed matrix of X. Orthogonal transformations preserve the Euclidean structure in $\mathbb{R}^n$.

Since $1 = \det(XX^{\mathrm{T}}) = \det^2 X$, it follows that orthogonal matrices have determinant $\det X = \pm 1$.

Example 2.4 (Special Orthogonal Group). Orthogonal unimodular matrices form the *special orthogonal group*:

$$\mathrm{SO}(n) = \{X \in M(n) \mid XX^{\mathrm{T}} = \mathrm{Id},\ \det X = 1\}.$$

Special orthogonal transformations preserve both the Euclidean structure and orientation in $\mathbb{R}^n$.

Example 2.5 (Affine Group). The *affine group* is defined as follows:

$$\mathrm{Aff}(n) = \left\{ X = \begin{pmatrix} Y & b \\ 0 & 1 \end{pmatrix} \in M(n+1) \mid Y \in \mathrm{GL}(n),\ b \in \mathbb{R}^n \right\} \subset \mathrm{GL}(n+1).$$

Such matrices correspond to affine transformations in $\mathbb{R}^n$: a linear mapping Y plus a translation b.

Example 2.6 (Euclidean Group). The *Euclidean group* is the following subgroup of the affine group:

$$\mathrm{E}(n) = \left\{ X = \begin{pmatrix} Y & b \\ 0 & 1 \end{pmatrix} \in M(n+1) \mid Y \in \mathrm{SO}(n),\ b \in \mathbb{R}^n \right\} \subset \mathrm{GL}(n+1).$$

Such matrices parametrize orientation-preserving linear isometries in $\mathbb{R}^n$.

Example 2.7 (Triangular Group). And the last example of a group formed by real matrices: the *triangular group* consists of all invertible triangular matrices:

$$\begin{aligned} \mathrm{T}(n) &= \left\{ X = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix} \in \mathrm{GL}(n) \right\} \\ &= \{X = (x_{ij}) \in M(n) \mid x_{ij} = 0,\ i > j,\ x_{ii} \neq 0\}. \end{aligned}$$

These are matrices of invertible linear operators preserving the flag of subspaces $\mathbb{R}e_1 \subset \mathrm{span}(e_1, e_2) \subset \cdots \subset \mathrm{span}(e_1, \ldots, e_{n-1}) \subset \mathbb{R}^n$.

Now we pass to complex matrices. Denote the space of all $n \times n$ matrices with complex entries as

$$M(n, \mathbb{C}) = \{Z = (z_{jk}) \mid z_{jk} \in \mathbb{C},\ j, k = 1, \ldots, n\}.$$

Since any entry decomposes into the real and imaginary parts:

$$z_{jk} = x_{jk} + iy_{jk},$$

a complex matrix decomposes correspondingly:

$$Z = X + iY, \qquad X = (x_{jk}), \quad Y = (y_{jk}).$$

The real coordinates x_{jk}, y_{jk} turn $M(n, \mathbb{C})$ into $\mathbb{R}^{2n^2}$.

The *realification* of a complex matrix is defined as follows. To any complex matrix of order n corresponds a real matrix of order $2n$:

$$Z \sim \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in M(2n, \mathbb{R}), \qquad Z = X + iY \in M(n, \mathbb{C}).$$

The matrix $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$ is just the matrix of the real linear operator in $\mathbb{R}^{2n} = \mathbb{C}^n$ corresponding to Z in the basis over reals $e_1, \dots, e_1, ie_1, \dots, ie_n$. So it is natural that realification respects the matrix product: if

$$Z_1 = X_1 + iY_1 \sim \begin{pmatrix} X_1 & -Y_1 \\ Y_1 & X_1 \end{pmatrix}, \qquad Z_2 = X_2 + iY_2 \sim \begin{pmatrix} X_2 & -Y_2 \\ Y_2 & X_2 \end{pmatrix},$$

then

$$\begin{aligned} Z_1 Z_2 &= X_1X_2 - Y_1Y_2 + i(X_2Y_1 + X_1Y_2) \\ &\sim \begin{pmatrix} X_1X_2 - Y_1Y_2 & -X_1Y_2 - X_2Y_1 \\ X_1Y_2 + X_2Y_1 & X_1X_2 - Y_1Y_2 \end{pmatrix} \\ &= \begin{pmatrix} X_1 & -Y_1 \\ Y_1 & X_1 \end{pmatrix} \cdot \begin{pmatrix} X_2 & -Y_2 \\ Y_2 & X_2 \end{pmatrix}. \end{aligned}$$

The realification provides the embedding $M(n, \mathbb{C}) \subset M(2n, \mathbb{R})$.

Example 2.8 (Complex General Linear Group). The *complex general linear group* consists of all complex $n \times n$ invertible matrices:

$$\begin{aligned} \mathrm{GL}(n, \mathbb{C}) &= \{Z \in M(n, \mathbb{C}) \mid \det Z \neq 0\} \\ &= \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in M(2n, \mathbb{R}) \mid \det^2 X + \det^2 Y \neq 0 \right\}. \end{aligned}$$

There holds a proposition similar to Theorem 2.1.

Theorem 2.2. *If G is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$, then G is a linear Lie group.*

Example 2.9 (Complex Special Linear Group). The *complex special linear group* is formed by all complex $n \times n$ unimodular matrices:

$$\begin{aligned} \mathrm{SL}(n, \mathbb{C}) &= \{Z \in M(n, \mathbb{C}) \mid \det Z = 1\} \\ &= \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in M(2n, \mathbb{R}) \mid \det(X + iY) = 1 \right\}. \end{aligned}$$

Example 2.10 (Unitary Group). An important example of a linear Lie group is the *unitary group* consisting of all $n \times n$ unitary matrices:

$$\mathrm{U}(n) = \{Z \in M(n, \mathbb{C}) \mid \det \bar{Z}^{\mathrm{T}} Z = \mathrm{Id}\}.$$

(Here $\bar{Z}$ denotes the complex conjugate matrix of Z.) Such matrices correspond to linear transformations that preserve the unitary structure in $\mathbb{C}^n$. Compute the realification of a unitary matrix.

We have

$$Z = X + iY, \qquad \bar{Z}^{\mathrm{T}} = X^{\mathrm{T}} - iY^{\mathrm{T}},$$

thus

$$\bar{Z}^{\mathrm{T}} Z \sim \begin{pmatrix} X^{\mathrm{T}} & Y^{\mathrm{T}} \\ -Y^{\mathrm{T}} & X^{\mathrm{T}} \end{pmatrix} \cdot \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$
$$= \begin{pmatrix} X^{\mathrm{T}}X + Y^{\mathrm{T}}Y & -X^{\mathrm{T}}Y + Y^{\mathrm{T}}X \\ -Y^{\mathrm{T}}X + X^{\mathrm{T}}Y & X^{\mathrm{T}}X + Y^{\mathrm{T}}Y \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \sim \mathrm{Id} + i \cdot 0.$$

That is why the realification of the unitary group reads

$$\mathrm{U}(n) = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in M(2n, \mathbb{R}) \mid X^{\mathrm{T}}X + Y^{\mathrm{T}}Y = \mathrm{Id},\ X^{\mathrm{T}}Y - Y^{\mathrm{T}}X = 0 \right\}.$$

Compute determinant of a unitary matrix:

$$1 = \det(\bar{Z}^{\mathrm{T}} Z) = \overline{\det Z} \cdot \det Z = |\det Z|^2,$$

so

$$\det Z = e^{i\varphi}, \qquad \varphi \in \mathbb{R}.$$

Example 2.11 (Special Unitary Group). Another important example of a group formed by complex matrices is the *special unitary group*:

$$\begin{aligned} \mathrm{SU}(n) &= \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C}) = \{Z \in M(n, \mathbb{C}) \mid \bar{Z}^{\mathrm{T}} Z = \mathrm{Id},\ \det Z = 1\} \\ &= \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in M(2n, \mathbb{R}) \mid X^{\mathrm{T}}X + Y^{\mathrm{T}}Y = \mathrm{Id},\ X^{\mathrm{T}}Y - Y^{\mathrm{T}}X = 0, \right. \\ &\qquad\qquad \left. \det(X + iY) = 1 \right\}. \end{aligned}$$

2.2 The Lie algebra of a Lie group

Example 2.12 ($T_{\mathrm{Id}}\,\mathrm{GL}(n)$). Consider the tangent space to the general linear group at the identity:

$$T_{\mathrm{Id}}\,\mathrm{GL}(n) = \left\{ \dot{X}(0) \mid X(t) \in \mathrm{GL}(n),\ X(0) = \mathrm{Id} \right\}.$$

We compute this space explicitly. Since the velocity vector $\dot{X}(t) = (\dot{x}_{ij}(t))$ is an $n \times n$ matrix, we obtain

$$\dot{X}(0) = (\dot{x}_{ij}(0)) = A \in M(n).$$

Thus

$$T_{\mathrm{Id}}\,\mathrm{GL}(n) \subset M(n).$$

In order to show that this inclusion is in fact an equality, choose an arbitrary matrix $A \in M(n)$. The curve $X(t) = \mathrm{Id} + tA$ belongs to $\mathrm{GL}(n)$ for $|t| < \varepsilon$ and small $\varepsilon > 0$. Moreover, $X(0) = \mathrm{Id}$ and $\dot{X}(0) = A$. Consequently,

$$T_{\mathrm{Id}}\,\mathrm{GL}(n) = M(n).$$

The tangent space $T_{\mathrm{Id}}\,\mathrm{GL}(n)$ is a linear space. Moreover, it is endowed with an additional operation, *commutator* of matrices:

$$[A, B] = AB - BA \in M(n), \qquad A,\ B \in M(n).$$

Notice that this operation satisfies the following properties:

(1) bilinearity,

(2) skew-symmetry: $[B, A] = -[A, B]$,

(3) *Jacobi identity*:

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0.$$

Definition 2.3. A linear space L endowed with a binary operation $[\,\cdot\,,\,\cdot\,]$ which is:

(1) bilinear,

(2) skew-symmetric, and

(3) satisfies Jacobi identity,

is called a *Lie algebra*.

The space $M(n)$ with the matrix commutator is a Lie algebra. In order to underline the relation of this Lie algebra with the Lie group $\mathrm{GL}(n)$, this Lie algebra is denoted as $\mathrm{gl}(n)$. Summing up,

$$T_{\mathrm{Id}}\,\mathrm{GL}(n) = \mathrm{gl}(n).$$

Such a construction has a generalization of fundamental importance.

Definition 2.4. The tangent space to a Lie group G at the identity element is called the *Lie algebra of the Lie group G*:

$$L = T_{\mathrm{Id}}G.$$

We compute the Lie algebras of the Lie groups considered above.

Example 2.13 (The Lie Algebra of $\mathrm{SL}(n)$). The Lie algebra of the special linear group is denoted by $\mathrm{sl}(n)$. We have

$$\mathrm{sl}(n) = T_{\mathrm{Id}}\,\mathrm{SL}(n) = \{\dot{X}(0) \mid X(t) \in \mathrm{SL}(n),\ X(0) = \mathrm{Id}\}.$$

Since a curve $X(t) \in \mathrm{SL}(n)$ satisfies the identity $\det X(t) \equiv 1$ and derivative of determinant is the trace of derivative, we get

$$0 = \frac{d}{dt} \det X(t) = \operatorname{tr} \frac{d}{dt} X(t) = \operatorname{tr} \dot{X}(t),$$

in particular

$$\operatorname{tr} \dot{X}(0) = 0.$$

Thus

$$\mathrm{sl}(n) = \{A \in M(n) \mid \operatorname{tr} A = 0\},$$

the traceless matrices. To be precise, we proved only the inclusion $\subset$. The reverse inclusion we do not prove here for the sake of time and leave it to the reader as an exercise. This can be done by comparing dimensions of the linear spaces. (The same remark holds for similar computations in examples below.)

Example 2.14 (The Lie Algebra of $\mathrm{SO}(n)$). This Lie algebra is denoted as

$$\mathrm{so}(n) = T_{\mathrm{Id}} \, \mathrm{SO}(n) = \{\dot{X}(0) \mid X(t) \in \mathrm{SO}(n),\ X(0) = \mathrm{Id}\}.$$

We have $X(t)X^{\mathrm{T}}(t) \equiv \mathrm{Id}$, thus

$$0 = \dot{X}(0) \underbrace{X^{\mathrm{T}}(0)}_{\mathrm{Id}} + \underbrace{X(0)}_{\mathrm{Id}} \dot{X}^{\mathrm{T}}(0) = \dot{X}(0) + \dot{X}^{\mathrm{T}}(0).$$

Denoting $A = \dot{X}(0)$, we obtain $A + A^{\mathrm{T}} = 0$ and

$$\mathrm{so}(n) = \{A \in M(n) \mid A + A^{\mathrm{T}} = 0\},$$

the skew-symmetric matrices.

In a similar way one computes the Lie algebras in the following three cases.

Example 2.15 (The Lie Algebra of $\mathrm{Aff}(n)$).

$$\mathrm{aff}(n) = \left\{ \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} \mid A \in \mathrm{gl}(n),\ b \in \mathbb{R}^n \right\}.$$

Example 2.16 (The Lie Algebra of $\mathrm{E}(n)$).

$$\mathrm{e}(n) = \left\{ \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} \mid A \in \mathrm{so}(n),\ b \in \mathbb{R}^n \right\}.$$

Example 2.17 (The Lie Algebra of $\mathrm{T}(n)$).

$$\mathrm{t}(n) = \{A = (a_{ij}) \in M(n) \mid a_{ij} = 0,\ i > j\},$$

the triangular matrices.

Finally compute the Lie algebras of the unitary and the special unitary groups.

Example 2.18 (The Lie Algebra of $\mathrm{U}(n)$).

$$\mathrm{u}(n) = T_{\mathrm{Id}} \, \mathrm{U}(n),$$

and we proceed in the same way as for $\mathrm{SO}(n)$. For a curve $Z(t) \in \mathrm{U}(n)$, $Z(0) = \mathrm{Id}$, we have $Z(t)\bar{Z}^{\mathrm{T}}(t) \equiv \mathrm{Id}$. Thus

$$0 = \dot{Z}(0)\underbrace{\bar{Z}^{\mathrm{T}}(0)}_{\mathrm{Id}} + \underbrace{Z(0)}_{\mathrm{Id}}\dot{\bar{Z}}^{\mathrm{T}}(0).$$

Denoting $A = \dot{Z}(0)$, we obtain $A + \bar{A}^{\mathrm{T}} = 0$, a skew-Hermitian matrix. Consequently,

$$\mathrm{u}(n) = \{A \in M(n,\mathbb{C}) \mid A + \bar{A}^{\mathrm{T}} = 0\}.$$

Example 2.19 (The Lie Algebra of $\mathrm{SU}(n)$).

$$\mathrm{su}(n) = \{A \in M(n,\mathbb{C}) \mid A + \bar{A}^{\mathrm{T}} = 0,\ \operatorname{tr} A = 0\}.$$

2.3 Matrix exponential

In order to approach matrix control systems, first consider a matrix ODE:

$$\dot{X} = AX, \tag{3}$$

where $A \in M(n)$ is a given matrix. In the case $n = 1$, solutions to the ODE

$$\dot{x} = ax$$

are given by the exponential:

$$\begin{aligned} &x(t) = e^{at}x(0), \\ &e^a = 1 + a + \frac{a^2}{2!} + \cdots + \frac{a^n}{n!} + \cdots . \end{aligned}$$

For arbitrary natural n, we can proceed in a similar way and define for a matrix $A \in M(n)$ its *exponential* by the same series:

$$\exp(A) = e^A = \mathrm{Id} + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots .$$

This matrix series converges absolutely, thus it can be differentiated termwise:

$$\begin{aligned} \left(e^{At}\right)' &= \left(\mathrm{Id} + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^nt^n}{n!} + \cdots\right)' \\ &= A + \frac{A^2t}{1!} + \cdots + \frac{A^nt^{n-1}}{n!} + \cdots \\ &= Ae^{At}. \end{aligned}$$

Thus the matrix exponential $X(t) = e^{At}$ is the solution to the Cauchy problem

$$\dot{X} = AX, \qquad X(0) = \mathrm{Id},$$

and all solutions of the matrix equation (3) have the form

$$X(t) = e^{At} X(0).$$

Notice that for an arbitrary matrix $A \in \mathrm{gl}(n)$, its exponential $\exp(A) \in \mathrm{GL}(n)$ since $\det \exp(A) = \exp(\operatorname{tr} A) \neq 0$. So we constructed a (smooth) mapping

$$\exp : \mathrm{gl}(n) \to \mathrm{GL}(n).$$

We generalize this construction in the following subsection.

2.4 Right-invariant vector fields

We saw that for an arbitrary matrix $A \in \mathrm{gl}(n)$, the Cauchy problem

$$\dot{X} = AX, \quad X(0) = X_0, \qquad X \in \mathrm{GL}(n),$$

has a (unique) solution of the form

$$X(t) = \exp(tA) X_0.$$

What about a similar problem in any (linear) Lie group G:

$$\dot{X} = AX, \qquad X \in G \ ?$$

Example 2.20. Consider e.g. a Cauchy problem in the special linear group:

$$\dot{X} = AX, \quad X(0) = \mathrm{Id}, \qquad X \in \mathrm{SL}(n). \tag{4}$$

By uniqueness, solutions of this ODE must be given, as above, by the matrix exponential, but the question is whether it is in the Lie group under consideration:

$$X(t) = \exp(tA) \in \mathrm{SL}(n) \ ?$$

It is obvious that in general the answer is negative:

$$\det \exp(tA) = \exp(t \operatorname{tr} A) \not\equiv 1 \text{ if } \operatorname{tr} A \neq 0.$$

In other words,

$$X(t) = \exp(tA) \in \mathrm{SL}(n) \ \Leftrightarrow \ A \in \mathrm{sl}(n).$$

If $A \notin \mathrm{sl}(n)$, then ODE (4) is not well-defined, i.e., the vector field AX is not tangent to the Lie group $\mathrm{SL}(n)$.

What is the tangent space to a Lie group G at its point X ? This question has a simple answer given in the following statement.

Proposition 2.1. *Let G be a linear Lie group, L its Lie algebra, and let $X \in G$. Then*

$$T_X G = T_{\mathrm{Id}} G \, X = LX = \{AX \mid A \in L\}.$$

Proof. Compute the tangent space

$$T_X G = \{\dot{X}(0) \mid X(t) \in G,\ X(0) = X\}.$$

For a smooth curve $X(t)$ starting from X, one easily constructs a curve starting from the identity:

$$Y(t) = X(t)X^{-1}, \qquad Y(0) = XX^{-1} = \mathrm{Id}\,.$$

Then

$$\dot{Y}(0) = \dot{X}(0)X^{-1} \in L.$$

We denote $A = \dot{Y}(0) \in L$ and get

$$\dot{X}(0) = AX, \qquad A \in L.$$

Thus $T_X G \subset LX$. Since these linear spaces have the same dimension, we obtain

$$T_X G = LX.$$

□

So the right product by X translates the tangent space L at identity to the tangent space LX at the point X.

Thus for any element

$$A \in L,$$

the vector

$$V(X) = AX \in T_X G, \qquad X \in G,$$

i.e., the vector field $V(X)$ is tangent to the Lie group G. And the ODE

$$\dot{X} = AX, \qquad X \in G, \tag{5}$$

is well-defined and has the solutions

$$X(t) = \exp(At)X(0) \in G.$$

Notice the following important property of ODE (5): if a curve $X(t)$ is a trajectory of the field $V(X) = AX$, then its right translation $X(t)Y$ is also a trajectory of this ODE for any $Y \in G$. Indeed:

$$X(t) = \exp(tA)X(0),$$

thus

$$Y(t) = X(t)Y = \exp(tA)X(0)Y = \exp(tA)Y(0).$$

Definition 2.5. Vector fields of the form

$$V(X) = AX, \quad X \in G, \qquad A \in L,$$

are called *right-invariant vector fields* on the linear Lie group G.

Suppose we have two right-invariant vector fields on a Lie group G:

$$\begin{aligned} &A,\ B \in L, \\ &V(X) = AX, \quad W(X) = BX, \qquad X \in G. \end{aligned}$$

There arises a natural question: what is the Lie bracket of such vector fields?

Recall one of the versions of the definition of Lie bracket of vector fields.

Definition 2.6. Let V and W be smooth vector fields on a smooth manifold M. The *Lie bracket* (or *commutator*) of the fields V, W is the vector field $[V, W] \in \operatorname{Vec} M$ such that

$$[V, W](X) = \left.\frac{d}{dt}\right|_{t=0} \gamma(\sqrt{t}), \qquad X \in M.$$

where the curve γ is defined as follows:

$$\gamma(t) = e^{-tV} \circ e^{-tW} \circ e^{tV} \circ e^{tW}(X).$$

Here e^{tV} denotes the flow of the vector field V:

$$\frac{d}{dt} e^{tV}(X) = V(e^{tV}(X)), \qquad \left. e^{tV}\right|_{t=0}(X) = X,$$

and $\operatorname{Vec} M$ denotes the set of all smooth vector fields on a smooth manifold M.

The Lie bracket of right-invariant vector fields is a right-invariant vector field.

Proposition 2.2. *Let G be a linear Lie group, L its Lie algebra, and let $A, B \in L$. Then*

$$[AX, BX] = [A, B]X = (AB - BA)X, \qquad X \in G.$$

Proof. For the right-invariant vector fields

$$V(X) = AX, \qquad W(X) = BX,$$

their flows are given by the matrix exponential:

$$e^{tV}(X) = \exp(tA)X, \qquad e^{tW}(X) = \exp(tB)X.$$

Compute the low-order terms of the curve γ from Definition 2.6:

$$\begin{aligned} \gamma(t) &= \exp(-tA)\exp(-tB)\exp(tA)\exp(tB)X \\ &= \left(\operatorname{Id} - tA + \frac{t^2}{2}A^2 + \cdots\right)\left(\operatorname{Id} - tB + \frac{t^2}{2}B^2 + \cdots\right) \\ &\qquad \left(\operatorname{Id} + tA + \frac{t^2}{2}A^2 + \cdots\right)\left(\operatorname{Id} + tB + \frac{t^2}{2}B^2 + \cdots\right)X \\ &= \left(\operatorname{Id} - t(A+B) + \frac{t^2}{2}(A^2 + 2AB + B^2) + \cdots\right) \\ &\qquad \left(\operatorname{Id} + t(A+B) + \frac{t^2}{2}(A^2 + 2AB + B^2) + \cdots\right)X \\ &= (\operatorname{Id} + t^2[A, B] + \cdots)X, \end{aligned}$$

thus

$$\gamma(\sqrt{t}) = (\mathrm{Id} + t[A, B] + \cdots)X,$$

notice that it is a smooth curve at $t = 0$,

$$\left.\frac{d}{dt}\right|_{t=0} \gamma(\sqrt{t}) = [A, B]X.$$

By Definition 2.6, this is the Lie bracket $[AX, BX]$. □

Corollary 2.1. *Right-invariant vector fields on a Lie group G form a Lie algebra isomorphic to the Lie algebra $L = T_{\mathrm{Id}}G$. The isomorphism is defined as follows:*

$$\textit{right-invariant vector field } AX \in \operatorname{Vec} G \qquad \leftrightarrow \qquad A \in L.$$

Thus in the sequel we identify these two representations of the Lie algebra of a Lie group G:

(1) $L = T_{\mathrm{Id}}G$, and

(2) $L = \{$right-invariant vector fields on $G\}$.

3 Right-invariant control systems

3.1 Definitions

Let G be a Lie group and L its Lie algebra.

Definition 3.1. A *right-invariant control system* Γ on a Lie group G is an arbitrary set of right-invariant vector fields on G, i.e., any subset

$$\Gamma \subset L. \tag{6}$$

Example 3.1 (Control-affine right-invariant systems). A particular class of right-invariant systems, which is important for applications is formed by *control-affine systems*

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u = (u_1, \ldots, u_m) \in U \subset \mathbb{R}^m \right\}, \tag{7}$$

where A, B_1, …, B_m are some elements of L. If the control set U coincides with $\mathbb{R}^m$, then system (7) is an affine subspace of L.

Remark. Throughout these notes, we write a right-invariant control system as (6) or (7), i.e., as a set of vector fields, a *polysystem*. In the *classical notation*, control-affine systems (7) are written as follows:

$$\dot{X} = AX + \sum_{i=1}^{m} u_i B_i X, \qquad u = (u_1, \ldots, u_m) \in U, \quad X \in G. \tag{8}$$

Polysystem (6) can also be written in such classical notation via a choice of a parametrization of the set Γ.

Definition 3.2. A *trajectory* of a right-invariant system Γ on G is a continuous curve $X(t)$ in G defined on an interval $[t_0, T] \subset \mathbb{R}$ so that there exists a partition

$$t_0 < t_1 < \cdots < t_N = T$$

and right-invariant vector fields

$$A_1, \ldots, A_N \in \Gamma$$

such that the restriction of $X(t)$ to each open interval (t_{i-1}, t_i) is differentiable and

$$\dot{X}(t) = A_i X(t) \text{ for } t \in (t_{i-1}, t_i), \quad i = 1, \ldots, N.$$

In the classical notation, this corresponds to piecewise-constant admissible controls. In the study of global controllability for infinite time we can restrict ourselves by such a class of admissible controls.

Definition 3.3. For any $T \geq 0$ and any X in G, the *reachable set for time T* of a right-invariant system $\Gamma \subset L$ from the point X is the set $\mathcal{A}_\Gamma(X, T)$ of all points that can be reached from X in exactly T units of time:

$$\mathcal{A}_\Gamma(X, T) = \{X(T) \mid X(\cdot) \text{ a trajectory of } \Gamma,\ X(0) = X\}.$$

The *reachable set for time not greater than* $T \geq 0$ is defined as

$$\mathcal{A}_\Gamma(X, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{A}_\Gamma(X, t).$$

The *reachable* (or *attainable*) *set* of a system Γ from a point $X \in G$ is the set $\mathcal{A}_\Gamma(X)$ of all terminal points $X(T)$, $T \geq 0$, of all trajectories of Γ starting at X:

$$\mathcal{A}_\Gamma(X) = \{X(T) \mid X(\cdot) \text{ a trajectory of } \Gamma,\ X(0) = X,\ T \geq 0\} = \bigcup_{T \geq 0} \mathcal{A}_\Gamma(X, T).$$

If there is no ambiguity, in the sequel we denote the reachable sets $\mathcal{A}_\Gamma(X, T)$ and $\mathcal{A}_\Gamma(X)$ by $\mathcal{A}(X, T)$ and $\mathcal{A}(X)$, respectively.

Definition 3.4. A system $\Gamma \subset L$ is called *controllable* if, given any pair of points X_0 and X_1 in G, the point X_1 can be reached from X_0 along a trajectory of Γ for a nonnegative time:

$$X_1 \in \mathcal{A}(X_0) \text{ for any } X_0,\ X_1 \in G,$$

or in other words, if

$$\mathcal{A}(X) = G \text{ for any } X \in G.$$

In the control literature, this notion corresponds to *global controllability*, or *complete controllability*. Although, for right-invariant systems these properties are equivalent to *local controllability*, see Theorem 3.5 below.

Remark. The inversion

$$i \, : \, G \to G, \qquad i(X) = X^{-1}$$

induces an isomorphism between the Lie algebra of right-invariant vector fields AX on a linear Lie group G and the Lie algebra of left-invariant vector fields XA on G. Thus, all problems for *left-invariant* control systems, including controllability, are reduced to the study of right-invariant systems.

3.2 Basic properties of orbits and reachable sets

Lemma 3.1. *Let $A \in L$ and $X_0 \in G$. Then the Cauchy problem*

$$\dot{X} = AX, \qquad X(t_0) = X_0,$$

has the solution $X(t) = \exp((t - t_0)A)\, X_0$.

Due to this obvious lemma we can obtain a description of an endpoint of a trajectory via product of exponentials.

Lemma 3.2. *Let $X(t)$, $t \in [0, T]$, be a trajectory of a right-invariant system $\Gamma \subset L$ with $X(0) = X_0$. Then there exist $N \in \mathbb{N}$ and*

$$\tau_1, \ldots, \tau_N > 0, \qquad A_1, \ldots A_N \in \Gamma$$

such that

$$\begin{aligned} &X(T) = \exp(\tau_N A_N) \, \cdots \, \exp(\tau_1 A_1) X_0, \\ &\tau_1 + \cdots + \tau_N = T. \end{aligned}$$

Proof. By the definition of a trajectory, there exist $N \in \mathbb{N}$ and

$$0 = t_0 < t_1 < \cdots < t_N = T, \qquad A_1, \ldots, A_N \in \Gamma$$

such that $X(t)$ is continuous and

$$t \in (t_{i-1}, t_i) \quad \Rightarrow \quad \dot{X} = A_i X(t).$$

Consider the first interval:

$$t \in (0, t_1) \quad \Rightarrow \quad \dot{X} = A_1 X(t), \quad X(0) = X_0.$$

Thus

$$X(t) = \exp(A_1 t) X_0, \qquad X(t_1) = \exp(A_1 t_1) X_0.$$

Further,

$$t \in (t_1, t_2) \quad \Rightarrow \quad \dot{X} = A_2 X(t), \quad X(t_1) = \exp(A_1 t_1) X_0.$$

So

$$\begin{aligned} X(t) &= \exp((t - t_1) A_2) \exp(t_1 A_1) X_0, \\ X(t_2) &= \exp((t_2 - t_1) A_2) \exp(A_1 t_1) X_0 \\ &= \exp(\tau_2 A_2) \exp(A_1 \tau_1) X_0, \qquad \tau_1 = t_1, \quad \tau_2 = t_2 - t_1. \end{aligned}$$

We go on in such a way and finally obtain the required representation:

$$\begin{aligned} & X(t_N) = X(T) = \exp(\tau_N A_N) \cdots \exp(\tau_1 A_1) X_0, \\ & \tau_N = t_N - t_{N-1}, \quad \cdots, \quad \tau_2 = t_2 - t_1, \quad \tau_1 = t_1, \\ & \tau_N + \cdots + \tau_1 = t_N = T. \end{aligned}$$

□

Now we can obtain a description of attainable sets and derive their elementary properties.

Lemma 3.3. *Let $\Gamma \subset L$ be a right-invariant system, and let X be an arbitrary point of G. Then*

(1) $\mathcal{A}_\Gamma(X) = \{\exp(t_N A_N) \cdots \exp(t_1 A_1) X \mid A_i \in \Gamma,\ t_i > 0,\ N \geq 0\}$;

(2) $\mathcal{A}_\Gamma(X) = \mathcal{A}_\Gamma(\mathrm{Id}) X$;

(3) $\mathcal{A}_\Gamma(\mathrm{Id})$ is a subsemigroup of G;

(4) $\mathcal{A}_\Gamma(X)$ is an arcwise-connected subset of G.

Proof. Items (1) and (2) follow immediately from Lemma 3.2.

(3) Since

$$\mathcal{A}_\Gamma(\mathrm{Id}) = \{\exp(t_N A_N) \cdots \exp(t_1 A_1) \mid A_i \in \Gamma,\ t_i > 0,\ N \geq 0\},$$

then for any $X_1,\ X_2 \in \mathcal{A}_\Gamma(\mathrm{Id})$, the product $X_1 X_2 \in \mathcal{A}_\Gamma(\mathrm{Id})$.

(4) Any point in $\mathcal{A}_\Gamma(X)$ is connected with the initial point X by a trajectory $X(t)$. □

Definition 3.5. The *orbit of a system* Γ through a point $X \in G$ is the following subset of the Lie group G:

$$\mathcal{O}_\Gamma(X) = \{\exp(t_N A_N) \cdots \exp(t_1 A_1) X \mid A_i \in \Gamma,\ t_i \in \mathbb{R},\ N \geq 0\},$$

compare with the description of attainable set $\mathcal{A}_\Gamma(X)$ given in item (1) of Lemma 3.3.

Obviously,

$$\mathcal{A}_\Gamma(X) \subset \mathcal{O}_\Gamma(X).$$

In the orbit, one is allowed to move both forward and backward in time, while in the attainable set only the forward motion is allowed. The structure of orbits is simpler than that of attainable sets.

Lemma 3.4. *Let $\Gamma \subset L$ be a right-invariant system, and let X be an arbitrary point of G. Then*

(1) $\mathcal{O}_\Gamma(X) = \mathcal{O}_\Gamma(\mathrm{Id})X$*;*

(2) $\mathcal{O}_\Gamma(\mathrm{Id})$ *is the connected Lie subgroup of G with the Lie algebra* $\mathrm{Lie}(\Gamma)$*.*

Here and below we denote by $\mathrm{Lie}(\Gamma)$ the Lie algebra generated by Γ, i.e., the smallest Lie subalgebra of L containing Γ.

Proof. Item (1) is obvious.

(2) First of all, the orbit $\mathcal{O}_\Gamma(\mathrm{Id})$ is connected since any point in it is connected with the identity by a continuous curve provided by the definition of an orbit.

Further, it is easy to see that $\mathcal{O}_\Gamma(\mathrm{Id})$ is a subgroup of G. If $X,\ Y \in \mathcal{O}_\Gamma(\mathrm{Id})$, then $XY \in \mathcal{O}_\Gamma(\mathrm{Id})$ as a product of exponentials. If

$$X = \exp(t_N A_N) \cdots \exp(t_1 A_1) \in \mathcal{O}_\Gamma(\mathrm{Id}),$$

then

$$X^{-1} = \exp(-t_1 A_1) \cdots \exp(-t_N A_N) \in \mathcal{O}_\Gamma(\mathrm{Id}).$$

Finally, $\mathrm{Id} \in \mathcal{O}_\Gamma(\mathrm{Id})$.

It follows from the general Orbit Theorem (see [9], [2]) that $\mathcal{O}_\Gamma(\mathrm{Id}) \subset G$ is a smooth submanifold with the tangent space $T_{\mathrm{Id}}\mathcal{O}_\Gamma(\mathrm{Id}) = \mathrm{Lie}(\Gamma)$.

Then the orbit $\mathcal{O}_\Gamma(\mathrm{Id})$ is a Lie subgroup of G with the Lie algebra $\mathrm{Lie}(\Gamma)$, see [23]. □

Proposition 3.1. *A right-invariant system Γ is controllable iff $\mathcal{A}_\Gamma(\mathrm{Id}) = G$.*

Proof. By definition, Γ is controllable iff $\mathcal{A}_\Gamma(X) = G$ for any $X \in G$. Since $\mathcal{A}_\Gamma(X) = \mathcal{A}_\Gamma(\mathrm{Id})X$, controllability is equivalent to the identity $\mathcal{A}_\Gamma(\mathrm{Id}) = G$. □

That is why in the sequel we use the following short notation for the attainable set and orbit from the identity:

$$\mathcal{A}_\Gamma(\mathrm{Id}) = \mathcal{A}_\Gamma = \mathcal{A}, \qquad \mathcal{O}_\Gamma(\mathrm{Id}) = \mathcal{O}_\Gamma = \mathcal{O}.$$

Given any subset l of a vector space V, we denote by $\mathrm{span}(l)$ the vector subspace of V generated by l and by $\mathrm{co}(l)$ the positive convex cone generated by the set l.

We denote the topological closure and the interior of a set S by $\mathrm{cl}\, S$ and $\mathrm{int}\, S$, respectively.

3.3 Normal attainability

If a point $Y \in G$ is reachable (or attainable) from a point $X \in G$, then there exist elements $A_1, \dots, A_N \in \Gamma$ and $t = (t_1, \dots, t_N) \in \mathbb{R}^N$ with positive coordinates such that

$$Y = \exp(t_N A_N) \cdots \exp(t_1 A_1) X.$$

That is, the point Y is in the image of the mapping

$$F \; : \; (s_1, \dots, s_N) \mapsto \exp(s_N A_N) \cdots \exp(s_1 A_1) X, \qquad s_i > 0.$$

The following stronger notion turns out to be important in the study of topological properties of reachable sets and controllability.

Definition 3.6. A point $Y \in G$ is called *normally attainable* from a point $X \in G$ by Γ if there exist elements $A_1, \dots, A_N$ in Γ and $t \in \mathbb{R}^N$ with positive coordinates $t_1, \dots, t_N$ such that the mapping

$$F \; : \; \mathbb{R}^N \to G, \qquad F(t_1, \dots, t_N) = \exp(t_N A_N) \cdots \exp(t_1 A_1) X$$

satisfies the following conditions:

(i) $F(t) = Y$.

(ii) $\operatorname{rank} D_t F = \dim G$.

The point Y is said to be normally accessible from X by $A_1, \dots, A_N$.

Lemma 3.5. *If a point $Y \in G$ is normally attainable from $X \in G$ by Γ, then $X \in \operatorname{int} \mathcal{A}_\Gamma$.*

Proof. The mapping F is a local diffeomorphism near t. That is, there exists a neighborhood $V \ni t$ such that the restriction $F|_V$ is a diffeomorphism. Then the set $F(V)$ is open. On the other hand, for any $s = (s_1, \dots, s_N)$, $s_i > 0$, the point $F(s)$ is in $\mathcal{A}_\Gamma(X)$. Thus $F(V) \subset \mathcal{A}_\Gamma(X)$ is a neighborhood of the point $Y = F(t)$. □

Theorem 3.1 (Krener). *Let $\operatorname{Lie}(\Gamma) = L$. Then:*

(1) *In any neighborhood V of the identity $\mathrm{Id} \in G$, there are points normally attainable from Id by Γ;*

(2) *Consequently, for any neighborhood $V \ni \mathrm{Id}$, the intersection $\operatorname{int} \mathcal{A} \cap V$ is nonempty;*

(3) *In particular, the interior $\mathcal{A}$ is nonempty.*

Proof. We prove item (1) since items (2) and (3) follow from it.

Denote $n = \dim L = \dim \operatorname{Lie}(\Gamma)$. If $n = 0$, everything is clear. Assume that $n \geq 1$ and fix a neighborhood V of the identity Id.

There exists a nonzero element $A_1 \in \Gamma$, otherwise $\dim \operatorname{Lie}(\Gamma) = 0$. Consider the mapping

$$F_1 \ : \ s_1 \mapsto \exp(s_1 A_1), \qquad 0 < s_1 < \varepsilon_1,$$

for sufficiently small positive ε_1. We have

$$\left. \frac{d F_1}{d s_1} \right|_{s_1 = 0} = A_1 \neq 0,$$

thus $\operatorname{rank} D_{s_1} F_1 = 1$ for small s_1. The curve

$$M_1 = \{ F_1(s_1) \mid 0 < s_1 < \varepsilon_1 \}$$

is a smooth one-dimensional manifold contained in the neighborhood V for sufficiently small positive ε_1. If $n = 1$, then any point $X_1 \in M_1$ is normally attainable from Id by A_1.

If $n > 1$, there exist an element $A_2 \in \Gamma$ and a point $X_1 \in M_1$ as close to identity as we wish such that

$$A_2(X_1) \notin T_{X_1} M_1.$$

Otherwise $\operatorname{Lie}(\Gamma)(X_1) \subset T_{X_1} M_1$ for any $X_1 \in M_1$ and $\dim \operatorname{Lie}(\Gamma) \leq \dim M_1 = 1$, a contradiction. We have

$$X_1 = \exp(t_1^1 A_1) \text{ for some } t_1^1 > 0.$$

Consider the mapping

$$F_2 \ : \ (s_1, s_2) \mapsto \exp(s_2 A_2) \exp(t_1^1 + s_1) A_1, \qquad 0 < s_i < \varepsilon_i,$$

and compute its derivatives:

$$\left. \frac{\partial F_2}{\partial s_2} \right|_0 = A_2(X_1), \qquad \left. \frac{\partial F_2}{\partial s_1} \right|_0 = A_1(X_1).$$

For small $s > 0$ we have $\operatorname{rank} D_s F_2 = 2$, thus the set

$$M_2 = \{ F_2(s_1, s_2) \mid 0 < s_i < \varepsilon_i, \ i = 1, 2 \}$$

is a smooth two-dimensional manifold that belongs to V for sufficiently small positive ε_1 and ε_2. If $n = 2$, the theorem is proved, since in this case any point of M_2 is normally attainable from Id by A_1 and A_2.

If $n > 2$, we proceed in a similar manner. There exist $A_3 \in \Gamma$ and $X_2 \in M_2$ close to Id such that

$$A_3(X_2) \notin T_{X_2} M_2.$$

Otherwise $\operatorname{Lie}(\Gamma)(X_2) \subset T_{X_2} M_2$ for any $X_2 \in M_2$ and $\dim \operatorname{Lie}(\Gamma) \leq 2$, a contradiction. We have

$$X_2 = \exp(t_2^2 A_2) \exp(t_1^2 A_1) \text{ for some } t_i^2 > 0.$$

Consider the mapping

$$F_3 \ : \ (s_1, s_2, s_3) \mapsto \exp(s_3 A_3) \exp((t_2^2 + s_2) A_2) \exp((t_1^2 + s_1) A_1), \qquad 0 < s_i < \varepsilon_i,$$

and compute its derivatives:

$$\left. \frac{\partial F_3}{\partial s_3} \right|_0 = A_3(X_2), \qquad \left. \frac{\partial F_3}{\partial s_2} \right|_0 = A_2(X_1),$$
$$\left. \frac{\partial F_3}{\partial s_1} \right|_0 = \exp_*(t_2^2 A_2) A_1(\exp(t_1^2 A_1)).$$

It follows that

$$\operatorname{span} \left(\left. \frac{\partial F_3}{\partial s_2} \right|_0, \left. \frac{\partial F_3}{\partial s_1} \right|_0 \right) = T_{X_2} M_2.$$

Since the vector field A_3 is not tangent to the manifold M_2 at the point X_2, the differential $D_s F_3$ has rank 3 for small $s > 0$. Thus

$$M_3 = \{ F_3(s_1, s_2, s_3) \mid 0 < s_i < \varepsilon_i, \ i = 1, 2, 3 \}$$

is a smooth three-dimensional manifold belonging to V for sufficiently small positive ε_i. In the case $n = 3$, the theorem is proved, otherwise we proceed by induction.

As a result of the inductive construction, we find an element $A_n \in \Gamma$ and a point

$$X_{n-1} = \exp(t_{n-1}^{n-1} A_{n-1}) \cdots \exp(t_1^{n-1} A_1) \in M_{n-1}$$

sufficiently close to Id such that

$$A_n(X_{n-1}) \notin T_{X_{n-1}} M_{n-1}.$$

Then the mapping

$$\begin{aligned} F_n \ : \ & (s_1, \ldots, s_n) \\ & \mapsto \exp(s_n A_n) \exp((t_{n-1}^{n-1} + s_{n-1}) A_{n-1}) \cdots \exp((t_1^{n-1} + s_1) A_1), \\ & \qquad\qquad\qquad\qquad 0 < s_i < \varepsilon_i, \end{aligned}$$

is an immersion for small $s > 0$. Consequently, any point $X_n \in M_n = \operatorname{Im} F_n$ is normally attainable from Id. Moreover, X_n can be chosen as close to Id as we wish. $\square$

Definition 3.7. A system $\Gamma \subset L$ is said to have a *full rank* (or to satisfy the *Lie Algebra Rank Condition*) if

$$\operatorname{Lie}(\Gamma) = L.$$

Proposition 3.2. *Let $\Gamma \subset L$. Then:*

(1) $\operatorname{int}_{\mathcal{O}} \mathcal{A} \neq \emptyset$*;*

(2) *Moreover,* $\mathcal{A} \subset \operatorname{cl}\operatorname{int}_{\mathcal{O}} \mathcal{A}$.

We denote by $\operatorname{int}_{\mathcal{O}} S$ the interior of a subset S of the orbit $\mathcal{O}$ in the topology of $\mathcal{O}$.

Proof. (a) Assume first that the system Γ has full rank: $\operatorname{Lie}(\Gamma) = L$, then the orbit $\mathcal{O} \subset G$ is an open subset, and the relative interior w.r.t. $\mathcal{O}$ coincides with the interior in G. By Krener's theorem, $\operatorname{int} \mathcal{A} \neq \emptyset$, and item (1) of this proposition follows.

We prove item (2). Take any element $X \in \mathcal{A}$ and any its neighborhood $V \ni X$. Then the open set VX^{-1} is a neighborhood of the identity. By Krener's theorem, there exists a point $Y \in VX^{-1} \cap \operatorname{int} \mathcal{A}$, thus $YX \in V$. Further, since $Y \in \operatorname{int} \mathcal{A}$, there exists a neighborhood $W \ni Y$, $W \subset \mathcal{A}$. Then the open set WX is a neighborhood of the point YX, moreover, $WX \subset \mathcal{A}$. Finally, $YX \in \operatorname{int} \mathcal{A} \cap V$. Consequently, any neighborhood V of the point X contains points from $\operatorname{int} \mathcal{A}$, thus $X \in \operatorname{cl}\operatorname{int} \mathcal{A}$.

(b) If $\operatorname{Lie}(\Gamma) \neq L$, we consider the restriction of the system Γ to the orbit $\mathcal{O}$, a Lie subgroup of G with the Lie algebra $\operatorname{Lie}(\Gamma)$. The system Γ is full-rank on $\mathcal{O}$, thus the statement in the case (b) follows from the case (a). □

3.4 General controllability conditions

Theorem 3.2 (Connectedness Condition). *If* $\Gamma \subset L$ *is controllable on* G*, then the Lie group* G *is connected.*

Proof. The attainable set $\mathcal{A}$ is a connected subset of G. □

Thus there are no controllable systems on disconnected Lie groups, e.g. on $\mathrm{GL}(n)$ or $\mathrm{O}(n)$. Instead, one can study controllability on the connected components of these groups through identity:

$$\mathrm{GL}_+(n) = \{X \in M(n) \mid \det X > 0\}$$

or $\mathrm{SO}(n)$.

Theorem 3.3 (Rank Condition). *Let* $\Gamma \subset L$.

(1) *If* Γ *is controllable, then* $\operatorname{Lie}(\Gamma) = L$.

(2) $\operatorname{int} \mathcal{A} \neq \emptyset$ *if and only if* $\operatorname{Lie}(\Gamma) = L$.

Proof. (1) If Γ is controllable, then $\mathcal{A} = G$, the more so $\mathcal{O} = G$, thus $\operatorname{Lie}(\Gamma) = L$.

(2) By Krener's theorem, if $\operatorname{Lie}(\Gamma) = L$, then $\operatorname{int} \mathcal{A} \neq \emptyset$.

Conversely, let $\operatorname{Lie}(\Gamma) \neq L$. Then $\dim \mathcal{O} = \dim \operatorname{Lie}(\Gamma) < \dim L = \dim G$. Thus $\operatorname{int} \mathcal{O} = \emptyset$, the more so $\operatorname{int} \mathcal{A} = \emptyset$. □

Theorem 3.4 (Group Test). *A system* $\Gamma \subset L$ *is controllable on a Lie group* G *iff the following conditions hold:*

(1) G *is connected,*

(2) $\mathrm{Lie}(\Gamma) = L$,

(3) $\mathcal{A}$ *is a subgroup of* G.

Proof. The necessity is obvious, we prove sufficiency. If $\mathcal{A} \subset G$ is a subgroup, then for any element $X \in \mathcal{A}$, its inverse X^{-1} belongs to $\mathcal{A}$ as well. Recall the descriptions of the attainable set and orbit through identity:

$$\mathcal{A} = \{\exp(t_N A_N) \cdots \exp(t_1 A_1) \mid t_i \geq 0,\ A_i \in \Gamma\},$$
$$\mathcal{O} = \{\exp(\pm t_N A_N) \cdots \exp(\pm t_1 A_1) \mid t_i \geq 0,\ A_i \in \Gamma\}.$$

For any exponential $\exp(t_i A_i) \in \mathcal{A}$, the inverse

$$(\exp(t_i A_i))^{-1} = \exp(-t_i A_i) \in \mathcal{A},$$

thus the attainable set $\mathcal{A}$ coincides with the orbit $\mathcal{O}$. But $\mathcal{O} \subset G$ is a connected Lie subgroup with Lie algebra $\mathrm{Lie}(\Gamma) = L$. Then it follows that $\mathcal{O} = G$, see [23]. Thus $\mathcal{A} = \mathcal{O} = G$. □

The following local controllability property is, for general nonlinear systems, weaker then the global controllability property we consider in our course.

Definition 3.8. A control system is called *locally controllable* at a point X if

$$X \in \operatorname{int} \mathcal{A}(X).$$

But for right-invariant systems these two notions coinside one with another (under the necessary connectedness condition).

Theorem 3.5 (Local Controllability Test). *A system* $\Gamma \subset L$ *is controllable on a Lie group* G *iff the following conditions hold:*

(1) G *is connected,*

(2) Γ *is locally controllable at the identity.*

Notice that identity element is always contained in the attainable set, and there may be two cases: either $\mathrm{Id} \in \operatorname{int} \mathcal{A}$, or $\mathrm{Id} \in \partial \mathcal{A}$. In the first case the system is controllable, while in the second case not.

Now we prove Theorem 3.5.

Proof. The necessity is obvious, we prove sufficiency. There exists a neighborhood $V \ni \mathrm{Id}$ such that $V \subset \mathcal{A}$. Consider the powers of this neighborhood: $V^n \subset \mathcal{A}$ for any $n \in \mathbb{N}$. But the Lie group G is connected, thus it is generated by any neighborhood of identity [23]:

$$\bigcup_{n \in \mathbb{N}} V^n = G.$$

Then $\mathcal{A} \supset \bigcup_{n \in \mathbb{N}} V^n = G$, thus $\mathcal{A} = G$. □

Theorem 3.6 (Closure Test). *A system $\Gamma \subset L$ is controllable on a Lie group G iff the following conditions hold:*

(1) $\mathrm{Lie}(\Gamma) = L$,

(2) $\mathrm{cl}\,\mathcal{A} = G$.

Proof. The necessity is straightforward. Let us prove the sufficiency. Consider the time-reversed system

$$-\Gamma = \{-A \mid A \in \Gamma\}.$$

Trajectories of the system $-\Gamma$ are trajectories of the initial system Γ passed in the opposite direction, thus

$$\begin{aligned}
\mathcal{A}_{-\Gamma} &= \{\exp(-t_N A_N) \cdots \exp(-t_1 A_1) \mid t_i \geq 0,\ A_i \in \Gamma\} \\
&= \left\{(\exp(t_N A_N) \cdots \exp(t_1 A_1))^{-1} \mid t_i \geq 0,\ A_i \in \Gamma\right\} \\
&= \mathcal{A}_\Gamma^{-1}.
\end{aligned}$$

Since $\mathrm{Lie}(-\Gamma) = \mathrm{Lie}(\Gamma) = L$, it follows that $\mathrm{int}\,\mathcal{A}_{-\Gamma} \neq \emptyset$, thus there exists an open subset $V \subset \mathcal{A}_{-\Gamma}$. Further, by the hypothesis of this theorem, $\mathrm{cl}\,\mathcal{A}_\Gamma = G$, thus there exists a point $X \in \mathcal{A}_\Gamma \cap V \neq \emptyset$. We have $X \in V \subset \mathcal{A}_{-\Gamma} = \mathcal{A}_\Gamma^{-1}$, thus the open set $V^{-1} \subset \mathcal{A}_\Gamma$ is a neighborhood of the inverse X^{-1}. Consequently, the open set $V^{-1}X \subset \mathcal{A}_\Gamma$. But $\mathrm{Id} = X^{-1}X \in V^{-1}X \subset \mathcal{A}_\Gamma$, thus $\mathrm{Id} \in \mathrm{int}\,\mathcal{A}_\Gamma$, and the system Γ is controllable by Theorem 3.5. $\square$

The previous theorem has important far-reaching consequences. It means that in the study of controllability one can replace the attainable set $\mathcal{A}$ by its closure $\mathrm{cl}\,\mathcal{A}$. This idea gives rise to the powerful extension techniques described in the following section.

4 Extension techniques for right-invariant systems

4.1 Saturate

Definition 4.1. Let $\Gamma_1, \Gamma_2 \subset L$. The system Γ_1 is called *equivalent* to the system Γ_2: $\Gamma_1 \sim \Gamma_2$ if

$$\mathrm{cl}\,\mathcal{A}_{\Gamma_1} = \mathrm{cl}\,\mathcal{A}_{\Gamma_2}.$$

It is easy to show that not only the attainable set $\mathcal{A}$, but also its closure is a semigroup.

Lemma 4.1. *Let $\Gamma \subset L$. Then $\mathrm{cl}\,\mathcal{A}_\Gamma$ is a subsemigroup of G.*

Proof. Let $X, Y \in \mathrm{cl}\,\mathcal{A}_\Gamma$. Then there exist sequences

$$\{X_n\},\ \{Y_n\} \subset \mathcal{A}_\Gamma \text{ such that } X_n \to X,\ Y_n \to Y \text{ as } n \to \infty.$$

Then
$$\{X_n Y_n\} \subset \mathcal{A}_\Gamma \text{ and } X_n Y_n \to XY \text{ as } n \to \infty.$$
$\square$

Lemma 4.2. *If* $\Gamma_1 \sim \Gamma$ *and* $\Gamma_2 \sim \Gamma$, *then* $\Gamma_1 \cup \Gamma_2 \sim \Gamma$.

Proof. We have $\operatorname{cl} \mathcal{A}_{\Gamma_1} = \operatorname{cl} \mathcal{A}_{\Gamma_2} = \operatorname{cl} \mathcal{A}_\Gamma$. The inclusion
$$\operatorname{cl} \mathcal{A}_\Gamma \subset \operatorname{cl} \mathcal{A}_{\Gamma_1 \cup \Gamma_2} \tag{9}$$
is obvious in view of the the chain $\operatorname{cl} \mathcal{A}_\Gamma = \operatorname{cl} \mathcal{A}_{\Gamma_1} \subset \operatorname{cl} \mathcal{A}_{\Gamma_1 \cup \Gamma_2}$.

Now we prove the inclusion
$$\mathcal{A}_{\Gamma_1 \cup \Gamma_2} \subset \operatorname{cl} \mathcal{A}_\Gamma. \tag{10}$$
Take an arbitrary element
$$X = \exp(t_N A_N) \cdots \exp(t_1 A_1) \in \mathcal{A}_{\Gamma_1 \cup \Gamma_2}, \qquad t_i \geq 0, \quad A_i \in \Gamma_1 \cup \Gamma_2.$$
We have
$$\exp(t_i A_i) \in \mathcal{A}_{\Gamma_1} \cup \mathcal{A}_{\Gamma_2} \subset \operatorname{cl} \mathcal{A}_\Gamma,$$
thus by Lemma 4.1 it follows that $X \in \operatorname{cl} \mathcal{A}_\Gamma$. So inclusion (10) is proved, and $\operatorname{cl} \mathcal{A}_{\Gamma_1 \cup \Gamma_2} \subset \operatorname{cl} \mathcal{A}_\Gamma$. In view of inclusion (9), it follows that $\Gamma_1 \cup \Gamma_2 \sim \Gamma$. $\square$

The previous lemma allows one to unite equivalent systems. It is then natural to consider the union of all systems equivalent to a given one.

Definition 4.2. The *saturate* of a right-invariant system $\Gamma \subset L$ is the following system:
$$\operatorname{Sat}(\Gamma) = \cup\{\Gamma' \subset L \mid \Gamma' \sim \Gamma\}.$$

Proposition 4.1. (1) $\operatorname{Sat}(\Gamma) \sim \Gamma$.

(2) $\operatorname{Sat}(\Gamma) = \{A \in L \mid \exp(\mathbb{R}_+ A) \subset \operatorname{cl} \mathcal{A}_\Gamma\}$.

Item (1) means that the saturate of Γ is the largest right-invariant system on G equivalent to Γ, while item (2) describes $\operatorname{Sat}(\Gamma)$ as a kind of a tangent object to $\operatorname{cl} \mathcal{A}_\Gamma$ at the identity.

Proof. (1) Obviously, $\Gamma \sim \Gamma$, thus $\Gamma \subset \operatorname{Sat}(\Gamma)$, so $\operatorname{cl} \mathcal{A}_\Gamma \subset \operatorname{cl} \mathcal{A}_{\operatorname{Sat}(\Gamma)}$. In order to prove the inclusion
$$\mathcal{A}_{\operatorname{Sat}(\Gamma)} \subset \operatorname{cl} \mathcal{A}_\Gamma, \tag{11}$$
take any element
$$X = \exp(t_N A_N) \cdots \exp(t_1 A_1) \in \mathcal{A}_{\operatorname{Sat}(\Gamma)}, \qquad t_i > 0, \quad A_i \in \operatorname{Sat}(\Gamma).$$
Each element A_i is contained in a system $\Gamma_i \sim \Gamma$, thus $\exp(t_i A_i) \in \mathcal{A}_{\Gamma_i} \subset \operatorname{cl} \mathcal{A}_\Gamma$. By the semigroup property, $\operatorname{cl} \mathcal{A}_\Gamma \ni X$. Inclusion (11) and item (1) follow.

(2) Denote the system

$$\widehat{\Gamma} = \{A \in L \mid \exp(\mathbb{R}_+ A) \subset \operatorname{cl} \mathcal{A}_\Gamma\}.$$

First we prove the inclusion

$$\widehat{\Gamma} \subset \operatorname{Sat}(\Gamma). \tag{12}$$

We show that $\widehat{\Gamma} \sim \Gamma$. Consider the representation

$$\mathcal{A}_{\widehat{\Gamma}} = \{\exp(t_N A_N) \cdots \exp(t_1 A_1) \mid t_i > 0,\ A_i \in \widehat{\Gamma}\}.$$

Since all $\exp(t_i A_i) \in \operatorname{cl} \mathcal{A}_\Gamma$, it follows that $\mathcal{A}_{\widehat{\Gamma}} \subset \operatorname{cl} \mathcal{A}_\Gamma$. Moreover, since $\Gamma \subset \widehat{\Gamma}$, then $\mathcal{A}_\Gamma \subset \mathcal{A}_{\widehat{\Gamma}}$. Thus $\operatorname{cl} \mathcal{A}_{\widehat{\Gamma}} = \operatorname{cl} \mathcal{A}_\Gamma$, hence $\widehat{\Gamma} \sim \Gamma$. Inclusion (12) is proved.

In order to prove the reverse inclusion

$$\operatorname{Sat}(\Gamma) \subset \widehat{\Gamma}, \tag{13}$$

take any element $A \in \operatorname{Sat}(\Gamma)$. Then $A \in \Gamma' \sim \Gamma$. Thus $\exp(tA) \in \operatorname{cl} \mathcal{A}_\Gamma$, i.e., $A \in \widehat{\Gamma}$. Inclusion (13) follows. Taking into account inclusion (12), we obtain the required equality: $\operatorname{Sat}(\Gamma) = \widehat{\Gamma}$. □

Remark. Unfortunately, the saturate is not the appropriate tangent object to $\operatorname{cl} \mathcal{A}$ responsible for controllability: it is possible that $\operatorname{Sat}(\Gamma) = L$, and nevertheless Γ is not controllable.

Example 4.1 (Irrational winding of the torus). The torus is a two-dimensional Abelian Lie group:

$$G = \mathbb{T}^2 = S^1 \times S^1 = \{(x \mod 1, y \mod 1)\}.$$

Its Lie algebra is

$$L = T_{\mathrm{Id}} \mathbb{T}^2 = \mathbb{R}^2.$$

Consider the following right-invariant system on G:

$$\Gamma = \{A\}, \qquad A = (1, r), \quad r \in \mathbb{R} \setminus \mathbb{Q}.$$

The attainable set is the irrational winding of the torus:

$$\begin{aligned} &\mathcal{A} = \exp(\mathbb{R}_+ A) = \{(x \mod 1, kx \mod 1) \mid x \geq 0\} \neq \mathbb{T}^2, \\ &\operatorname{cl} \mathcal{A} = \mathbb{T}^2. \end{aligned}$$

Thus

$$\Gamma \sim L = \operatorname{Sat}(\Gamma),$$

although Γ is not controllable on $\mathbb{T}^2$. The reason is clear — the rank condition is violated:

$$\operatorname{Lie}(\Gamma) = \mathbb{R}A \neq L.$$

4.2 Lie saturate

It is the following Lie-generated tangent object to $\operatorname{cl}\mathcal{A}_\Gamma$ that is responsible for controllability of Γ.

Definition 4.3. *Lie saturate* of a right-invariant system is defined as follows:

$$\begin{aligned}\operatorname{LS}(\Gamma) &= \operatorname{Lie}(\Gamma) \cap \operatorname{Sat}(\Gamma) \\ &= \{A \in \operatorname{Lie}(\Gamma) \mid \exp(\mathbb{R}_+ A) \subset \operatorname{cl}\mathcal{A}_\Gamma\}.\end{aligned}$$

Theorem 4.1 (Lie Saturate Test). *A right-invariant system $\Gamma \subset L$ is controllable on a connected Lie group G if and only if $\operatorname{LS}(\Gamma) = L$.*

Proof. Necessity follows from the definition of the Lie saturate.

Sufficiency. Assume that $\operatorname{LS}(\Gamma) = L$. The connected Lie group G is generated by the one-parameter semigroups $\{\exp(tA) \mid A \in L,\ t \geq 0\}$ as a semigroup; thus the equality $\operatorname{Sat}(\Gamma) = L$ implies that $\operatorname{cl}(\mathcal{A}) = G$. Since, in addition, the rank condition $\operatorname{Lie}(\Gamma) = L$ holds, then Γ is controllable by Theorem 3.6. □

The basic properties of Lie saturate are collected in the following proposition.

Theorem 4.2. (1) $\operatorname{LS}(\Gamma)$ *is a closed convex positive cone in L, i.e.,*

(1a) $\operatorname{LS}(\Gamma)$ *is topologically closed*:

$$\operatorname{cl}(\operatorname{LS}(\Gamma)) = \operatorname{LS}(\Gamma),$$

(1b) $\operatorname{LS}(\Gamma)$ *is convex*:

$$A, B \in \operatorname{LS}(\Gamma) \ \Rightarrow\ \alpha A + (1-\alpha)B \in \operatorname{LS}(\Gamma) \quad \forall\, \alpha \in [0,1],$$

(1c) $\operatorname{LS}(\Gamma)$ *is a positive cone*:

$$A \in \operatorname{LS}(\Gamma) \ \Rightarrow\ \alpha A \in \operatorname{LS}(\Gamma) \quad \forall\, \alpha \geq 0.$$

Thus,

$$A, B \in \operatorname{LS}(\Gamma) \ \Rightarrow\ \alpha A + \beta B \in \operatorname{LS}(\Gamma) \quad \forall\, \alpha, \beta \geq 0.$$

(2) *For any $\pm A,\ B \in \operatorname{LS}(\Gamma)$ and any $s \in \mathbb{R}$,*

$$\begin{aligned}\exp(s \operatorname{ad} A)B = B + (s \operatorname{ad} A)B + \frac{(s \operatorname{ad} A)^2}{2!}B + \ldots + \frac{(s \operatorname{ad} A)^n}{n!}B + \cdots \\ \in \operatorname{LS}(\Gamma).\end{aligned}$$

(3) *If $\pm A, \pm B \in \operatorname{LS}(\Gamma)$, then $\pm[A, B] \in \operatorname{LS}(\Gamma)$.*

(4) *If $A \in \operatorname{LS}(\Gamma)$ and if the one-parameter subgroup $\{\exp(tA) \mid t \in \mathbb{R}\}$ is periodic (i.e., compact), then $-A \in \operatorname{LS}(\Gamma)$.*

(5) *Moreover, if* $A \in \mathrm{LS}(\Gamma)$ *and if the one-parameter subgroup* $\{\exp(tA) \mid t \in \mathbb{R}\}$ *is quasi-periodic:*

$$\exp(\mathbb{R}_- A) \subset \mathrm{cl} \exp(\mathbb{R}_+ A), \tag{14}$$

then $-A \in \mathrm{LS}(\Gamma)$.

We denote by $\mathrm{ad}\, A$ the adjoint operator corresponding to $A \in L$:

$$\mathrm{ad}\, A \; : \; L \to L, \qquad \mathrm{ad}\, A \; : \; B \mapsto [A, B].$$

Proof. (1a) Take a converging sequence $\mathrm{LS}(\Gamma) \ni A_n \to A \in L$. Since the linear space $\mathrm{Lie}(\Gamma)$ is closed, we have

$$A_n \in \mathrm{Lie}(\Gamma) \quad \Rightarrow \quad A \in \mathrm{Lie}(\Gamma).$$

Further, it follows that $\mathrm{Sat}(\Gamma)$ is closed as well: since $A_n \in \mathrm{Sat}(\Gamma)$, we have

$$\mathrm{cl}\, \mathcal{A} \ni \exp(tA_n) \to \exp(tA) \in \mathrm{cl}\, \mathcal{A}, \qquad t \geq 0,$$

and $A \in \mathrm{Sat}(\Gamma)$. Consequently, $\mathrm{LS}(\Gamma) = \mathrm{Lie}(\Gamma) \cap \mathrm{Sat}(\Gamma)$ is topologically closed.

(1b) Take any $A, B \in \mathrm{LS}(\Gamma)$, $\alpha \in [0, 1]$, and consider the convex combination $C = \alpha A + \beta B$, $\beta = 1 - \alpha$. It is easy to see (exercise) that

$$\exp(tC) = \lim_{n \to \infty} \underbrace{\exp\left(\frac{\alpha}{n} tA\right) \exp\left(\frac{\beta}{n} tB\right) \cdots \exp\left(\frac{\alpha}{n} tA\right) \exp\left(\frac{\beta}{n} tB\right)}_{n \text{ pairs}},$$

thus $C \in \mathrm{Sat}(\Gamma)$, and $\mathrm{Sat}(\Gamma)$ is convex. Since the linear space $\mathrm{Lie}(\Gamma)$ is convex, it follows that $\mathrm{LS}(\Gamma)$ is convex as well.

(1c) It is easy to show that $\mathrm{LS}(\Gamma)$ is a cone. Take any $A \in \mathrm{LS}(\Gamma)$, $\alpha > 0$. Then $\exp(t\alpha A) \in \mathrm{cl}\, \mathcal{A}$, $t \geq 0$, i.e, $\alpha A \in \mathrm{LS}(\Gamma)$.

To prove (2), assume that $\pm A, B \in \mathrm{LS}(\Gamma)$. Denote the element

$$B_s = \exp(s\, \mathrm{ad}\, A) B, \qquad s \in \mathbb{R}. \tag{15}$$

It is easy to see that this element admits the following representation (here we suppose that G is a linear Lie group):

$$B_s = \exp(sA) B \exp(-sA). \tag{16}$$

Indeed, the both curves (15) and (16) are solutions to the Cauchy problem

$$B_0 = B, \qquad \frac{d}{d\, s} B_s = [A, B_s].$$

Further, it is obvious from (15) that $B_s \in \mathrm{Lie}(\Gamma)$. Representation (16) implies that

$$\exp(tB_s) = \exp(tA) \exp(sB) \exp(-tA) \in \mathrm{cl}(\mathcal{A}_\Gamma)$$

for any $t \geq 0$, $s \in \mathbb{R}$; thus $B_s \in \mathrm{LS}(\Gamma)$ for all $s \in \mathbb{R}$.

Now (3) easily follows: if $\pm A, \pm B \in \mathrm{LS}(\Gamma)$, then $\pm e^{t \operatorname{ad} A} B, \pm B \in \mathrm{LS}(\Gamma)$, that is why

$$\pm[A, B] = \pm \lim_{t \to 0} \frac{e^{t \operatorname{ad} A} B - B}{t} \in \mathrm{LS}(\Gamma).$$

(4) follows from the chain

$$\{\exp(tA) \mid t \geq 0\} = \{\exp(tA) \mid t \in \mathbb{R}\} \subset \operatorname{cl} \mathcal{A}_\Gamma,$$

which is valid for all $A \in \mathrm{LS}(\Gamma)$ with a periodic one-parameter group.

Finally, we prove a more strong property (5). It follows from the quasi-periodic property (14) that

$$\exp(-tA) = \exp(t(-A)) \in \operatorname{cl} \exp(\mathbb{R}_+ A) \subset \operatorname{cl} \mathcal{A}_\Gamma$$

for any $t \geq 0$, thus $-A \in \mathrm{LS}(\Gamma)$. □

Usually, it is difficult to construct the Lie saturate of a right-invariant system explicitly. That is why Theorems 4.1 and 4.2 are applied as sufficient conditions of controllability via the following procedure. Starting from a given system Γ, one constructs a completely ordered ascending family of extensions $\{\Gamma_\alpha\}$ of Γ, i.e.,

$$\Gamma_0 = \Gamma, \qquad \Gamma_\alpha \subset \Gamma_\beta \text{ if } \alpha < \beta.$$

The extension rules are provided by Theorem 4.2:

(1) given Γ_α, one constructs $\Gamma_\beta = \operatorname{cl}(\operatorname{co}(\Gamma_\alpha))$;

(2) for $\pm A, B \in \Gamma_\alpha$, one constructs $\Gamma_\beta = \Gamma_\alpha \cup e^{\mathbb{R} \operatorname{ad} A} B$;

(3) for $\pm A, \pm B \in \Gamma_\alpha$, one constructs $\Gamma_\beta = \Gamma_\alpha \cup \mathbb{R}[A, B]$;

(4, 5) given $A \in \Gamma_\alpha$ with periodic or quasi-periodic one-parameter group, one constructs $\Gamma_\beta = \Gamma_\alpha \cup \mathbb{R}A$.

Theorem 4.2 guarantees that all extensions Γ_α belong to $\mathrm{LS}(\Gamma)$. If one obtains the relation $\Gamma_\alpha = L$ at some step α, then $\mathrm{LS}(\Gamma) = L$, and the system Γ is controllable by Theorem 4.1.

5 Induced systems on homogeneous spaces

Example 5.1 (Bilinear systems). Consider the following right-invariant system on $\mathrm{GL}_+(n)$:

$$\Gamma = A + \mathbb{R}B \subset \mathrm{gl}(n).$$

In the classical notation, this system reads

$$\dot{X} = AX + uBX, \qquad X \in \mathrm{GL}_+(n), \quad u \in \mathbb{R}. \tag{17}$$

Introduce also the following bilinear system:

$$\dot{x} = Ax + uBx, \qquad x \in \mathbb{R}^n \setminus \{0\}, \quad u \in \mathbb{R}. \tag{18}$$

We exclude the origin from $\mathbb{R}^n$ since linear vector fields vanish at the origin, thus it is an equilibrium for bilinear systems.

If $X(t)$ is a trajectory of the right-invariant system (17) with $X(0) = \mathrm{Id}$, then the curve $x(t) = X(t)x_0$ is a trajectory of the bilinear system (18) with $x(0) = x_0$.

Assume that the right-invariant system is controllable on $\mathrm{GL}_+(n)$. Then it is easy to see that the bilinear system is controllable on $\mathbb{R}^n \setminus \{0\}$. Indeed, take any two points $x_0,\ x_1 \in \mathbb{R}^n \setminus \{0\}$. There exists a matrix $X_1 \in \mathrm{GL}_+(n)$ such that $X_1 x_0 = x_1$. By virtue of controllability of Γ, there exists a trajectory $X(t)$ of the right-invariant system such that $X(0) = \mathrm{Id}$, $X(T) = X_1$ for some $T \geq 0$. Then the trajectory $x(t) = X(t)x_0$ of the bilinear system steers x_0 to x_1:

$$x(0) = X(0)x_0 = \mathrm{Id}\, x_0 = x_0, \qquad x(T) = X(T)x_0 = X_1 x_0 = x_1.$$

We showed that if the right-invariant system (17) is controllable on $\mathrm{GL}_+(n)$, then the bilinear system (18) is controllable on $\mathbb{R}^n \setminus \{0\}$.

There were three key points in the preceding argument.

(1) The Lie group $G = \mathrm{GL}_+(n)$ acts on the manifold $M = \mathbb{R}^n \setminus \{0\}$, that is, any $X \in G$ defines a mapping

$$X\ :\ M \to M, \qquad X\ :\ x \mapsto Xx.$$

(2) G acts transitively on M:

$$\forall\ x_0\ x_1 \in M \quad \exists\ X \in G \text{ such that } Xx_0 = x_1.$$

(3) The bilinear system (18) is induced by the right-invariant system (17): if $X(t)$ is a trajectory of (17), then $X(t)x$ is a trajectory of (18).

This construction generalizes as follows.

Definition 5.1. A Lie group G is said to *act on a smooth manifold* M if there exists a smooth mapping

$$\theta\ :\ G \times M \to M$$

that satisfies the following conditions:

(1) $\theta(YX, x) = \theta(Y, \theta(X, x))$ for any $X, Y \in G$ and any $x \in M$;

(2) $\theta(\mathrm{Id}, x) = x$ for any $x \in M$.

Definition 5.2. A Lie group G acts *transitively* on M if for any $x_0, x_1 \in M$ there exists $X \in G$ such that $\theta(X, x_0) = x_1$. A manifold that admits a transitive action of a Lie group is called the *homogeneous space* of this Lie group.

Definition 5.3. Let $A \in L$. The vector field $\theta_* A \in \operatorname{Vec} M$ *induced* by the action θ is defined as follows:

$$(\theta_* A)(x) = \left.\frac{d}{dt}\right|_{t=0} \theta(\exp(tA), x), \quad x \in M.$$

Definition 5.4. Let $\Gamma \subset L$ be a right-invariant system. The system

$$\begin{aligned}
&\theta_*\Gamma \subset \operatorname{Vec} M, \\
&(\theta_*\Gamma)(x) = \{(\theta_* A)(x) \mid A \in \Gamma\}, \qquad x \in M,
\end{aligned}$$

is called the *induced system* on M.

Lemma 5.1. *If $X(t)$ is a trajectory of Γ, then $x(t) = \theta(X(t), x_0)$ is a trajectory of $\theta_*\Gamma$ for any $x_0 \in M$.*

Proof. We can consider the case where the whole trajectory $X(t)$ satisfies a single ODE $\dot{X} = AX(t)$, $A \in \Gamma$, since an arbitrary trajectory of Γ is a concatenation of such pieces. Then $X(t) = \exp(At)X_0$ and $x(t) = \theta(\exp(At)X_0, x_0)$. Then the required ODE is verified by differentiation:

$$\begin{aligned}
\dot{x}(t) &= \frac{d}{dt}\theta(\exp(At)X_0, x_0) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \theta(\exp(A(t+\varepsilon))X_0, x_0) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \theta(\exp(A\varepsilon), \underbrace{\theta(\exp(At)X_0, x_0)}_{x(t)}) \\
&= (\theta_* A)(x(t)).
\end{aligned}$$

□

Theorem 5.1. *Let θ be a transitive action of a connected Lie group G on a manifold M, let $\Gamma \subset L$ be a right-invariant system on G, and let $\theta_*\Gamma \subset \operatorname{Vec} M$ be the induced system on M.*

(1) *If Γ is controllable on G, then $\theta_*\Gamma$ is controllable on M.*

(2) *Moreover, if the semigroup $\mathcal{A}_\Gamma$ acts transitively on M, then $\theta_*\Gamma$ is controllable on M.*

Proof. Item (1) follows from (2), so we prove (2). Take any points $x_0, x_1 \in M$. The transitivity of action of $\mathcal{A}_\Gamma$ on M means that there exists $X \in \mathcal{A}_\Gamma$ such that $\theta(X, x_0) = x_1$. Further, the inclusion $X \in \mathcal{A}_\Gamma$ means that some trajectory $X(t)$ of Γ steers Id to X: $X(0) = \mathrm{Id}$, $X(T) = X$, $T \geq 0$. Then the curve $x(t) = \theta(X(t), x_0)$ is a trajectory of $\theta_*\Gamma$ that steers x_0 to x_1:

$$x(0) = \theta(\mathrm{Id}, x_0) = x_0, \qquad x(T) = \theta(X, x_0) = x_1.$$

□

Important applications of Theorem 5.1 are related to the linear action of linear groups $G \subset \mathrm{GL}(n;\mathbb{R})$ on the vector space $\mathbb{R}^n$. In this case, the induced systems are bilinear, or more generally, affine systems.

Example 5.2 ($G = \mathrm{GL}_+(\mathbb{R})$, $M = \mathbb{R}^n \setminus \{0\}$). We have

$$\begin{aligned} &\theta(X, x) = Xx, \\ &\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \right\} \subset \mathrm{gl}(n), \\ &\theta_* \Gamma \ : \ \dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x, \qquad x \in \mathbb{R}^n \setminus \{0\}. \end{aligned}$$

If $\mathcal{A} = \mathrm{GL}_+(n)$ or $\mathcal{A} = \mathrm{SL}(n)$, then the bilinear system $\theta_*\Gamma$ is controllable on $\mathbb{R}^n \setminus \{0\}$. The attainable set may be even less, for example, in the case $\mathcal{A} = \mathrm{SO}(n) \times \mathbb{R}_+ \,\mathrm{Id}$ the bilinear system $\theta_*\Gamma$ remains controllable.

Remark. Linear groups acting transitively on $\mathbb{R}^n \setminus \{0\}$ or S^n are described, see [4, 5, 19, 6, 7, 14].

Example 5.3 ($G = \mathrm{SL}(n)$, $M = \mathbb{R}^n \setminus \{0\}$). Similarly,

$$\begin{aligned} &\theta(X, x) = Xx, \\ &\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \right\} \subset \mathrm{sl}(n), \\ &\theta_* \Gamma \ : \ \dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x, \qquad x \in \mathbb{R}^n \setminus \{0\}. \end{aligned}$$

If $\mathcal{A}$ is transitive on $\mathbb{R}^n \setminus \{0\}$, then the bilinear system $\theta_*\Gamma$ is controllable on $\mathbb{R}^n \setminus \{0\}$.

Example 5.4 ($G = \mathrm{SO}(n)$, $M = S^{n-1}$).

$$\begin{aligned} &\theta(X, x) = Xx, \\ &\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \right\} \subset \mathrm{so}(n), \\ &\theta_* \Gamma \ : \ \dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x, \qquad x \in S^{n-1}. \end{aligned}$$

Example 5.5 ($G = \mathrm{U}(n)$ or $\mathrm{SU}(n)$, $M = S^{2n-1}$).

$$\begin{aligned} &\theta(Z, z) = Zz, \\ &\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \right\} \subset \mathrm{u}(n) \text{ or } \mathrm{su}(n), \\ &\theta_* \Gamma \ : \ \dot{z} = Az + \sum_{i=1}^{m} u_i B_i z, \qquad z \in S^{2n-1}. \end{aligned}$$

Example 5.6 ($G = \mathrm{Aff}_+(n)$, $M = \mathbb{R}^n$). The connected component of identity in the affine group

$$\mathrm{Aff}_+(n) = \left\{ \begin{pmatrix} X & y \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}(n+1)$$

acts transitively on the space

$$M = \mathbb{R}^n = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^{n+1}$$

as follows:

$$\theta \left(\begin{pmatrix} X & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x \\ 1 \end{pmatrix} \right) = \begin{pmatrix} X & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = Xx + y.$$

Consider a right-invariant system on G:

$$\Gamma = \left\{ C_0 + \sum_{i=1}^m u_i C_i \right\}, \qquad C_i = \begin{pmatrix} A_i & b_i \\ 0 & 0 \end{pmatrix} \in \mathrm{aff}(n).$$

The induced vector fields are affine:

$$\theta_* \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = Ax + b,$$

and the induced system reads

$$\theta_*\Gamma \ : \ \dot{x} = A_0 x + b_0 + \sum_{i=1}^m u_i (A_i x + b_i), \qquad x \in \mathbb{R}^n.$$

In particular, for $b_0 = 0$, $A_1 = \cdots = A_m = 0$, we obtain the *linear system*

$$\dot{x} = A_0 x + \sum_{i=1}^m u_i b_i, \qquad x \in \mathbb{R}^n, \quad u_i \in \mathbb{R}. \tag{19}$$

Exercise 5.1. Show that if the *Kalman condition* holds:

$$\mathrm{span}(b_1, \dots, b_m;\ A_0 b_1, \dots, A_0 b_m;\ \dots;\ A_0^{n-1} b_1, \dots, A_0^{n-1} b_m) = \mathbb{R}^n,$$

then the linear system (19) is controllable on $\mathbb{R}^n$.

Example 5.7 ($G = \mathrm{E}(n)$, $M = \mathbb{R}^n$). This case is completely similar to the case of $\mathrm{Aff}_+(n)$.

6 Symmetric systems

Definition 6.1. A system $\Gamma \subset L$ is called *symmetric* if

$$\Gamma = -\Gamma,$$

i.e., together with any element A, this system contains also the sign-opposite element $-A$.

Given a symmetric system, for any admissible direction of motion A, the motion in the opposite direction $-A$ is also admissible.

Lemma 6.1. *Let* $\Gamma = -\Gamma$. *Then* $\mathcal{A} = \mathcal{O}$.

Proof. We have

$$\mathcal{O} = \{\exp(\pm t_N A_N) \cdots \exp(\pm t_1 A_1) \mid t_i > 0, \; A_i \in \Gamma\}.$$

But all $-A_i \in \Gamma$, thus $\mathcal{A} = \mathcal{O}$. □

Thus the study of controllability for symmetric Γ is reduced to the verification of the rank condition.

Theorem 6.1. *A symmetric right-invariant system* $\Gamma \subset L$ *is controllable on a connected Lie group* G *if and only if* $\mathrm{Lie}(\Gamma) = L$.

Proof. The necessity is a general fact. Sufficiency follows since for a full-rank system on a connected Lie group the orbit coincides with the whole Lie group. □

Example 6.1 (Control-linear systems). A *control-linear system*

$$\Gamma = \left\{ \sum_{i=1}^{m} u_i A_i \mid u = (u_1, \dots, u_m) \in U \subset \mathbb{R}^m \right\}$$

is symmetric if the set of control parameters U is symmetric with respect to the origin: $U = -U$; in particular, if $U = \mathbb{R}^m$:

$$\Gamma = \mathrm{span}(A_1, \dots, A_m) \subset L.$$

Such a system is controllable on a connected Lie group G iff $\mathrm{Lie}(A_1, \dots, A_m) = L$.

Example 6.2 (Symmetric bilinear system). Let $A_1, \dots, A_m \in \mathrm{gl}(n)$. Consider the corresponding symmetric bilinear system:

$$\dot{x} = \sum_{i=1}^{m} u_i A_i x, \qquad x \in \mathbb{R}^n \setminus \{0\}, \quad u_i \in \mathbb{R}. \tag{20}$$

Denote $\mathrm{Lie}(A_1, \dots, A_m) = L$, and let $G \subset \mathrm{GL}(n)$ be the connected Lie subgroup corresponding to the Lie algebra L. If G acts transitively on $\mathbb{R}^n \setminus \{0\}$ (or S^{n-1}), then the bilinear system (20) is controllable on $\mathbb{R}^n \setminus \{0\}$ (respectively on S^{n-1}).

7 Compact Lie groups

In this section, we consider the case of a Lie group that is *compact* as a topological space. For example, the Lie groups $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$ are compact.

The following simple fact is crucial for the controllability problem on compact Lie groups.

Lemma 7.1. *Let a Lie group G be compact, and let A belongs to the Lie algebra L. Then the one-parameter subgroup* $\exp(\mathbb{R}A)$ *is quasi-periodic:*

$$\exp(\mathbb{R}_- A) \subset \operatorname{cl} \exp(\mathbb{R}_+ A).$$

Proof. Denote $X = \exp(tA)$ for an arbitrary fixed $t > 0$. We have to prove that

$$\exp(-tA) = X^{-1} \in \operatorname{cl} \exp(\mathbb{R}_+ A).$$

The sequence $\{X^n\}$, $n \in \mathbb{N}$, has a converging subsequence in the compact Lie group G:

$$X^{n_k} \to Y \in G \text{ as } k \to \infty, \qquad n_{k+1} > n_k.$$

Then

$$X^{n_{k+1}-n_k-1} = X^{n_{k+1}} X^{-n_k} X^{-1} \to YY^{-1}X^{-1} = X^{-1} \text{ as } k \to \infty.$$

But $n_{k+1} - n_k - 1 \geq 0$, thus $X^{-1} \in \operatorname{cl} \exp(\mathbb{R}_+ A)$. □

Corollary 7.1. *Let G be compact, and let $\Gamma \subset L$. Then* $\mathrm{LS}(\Gamma) = \mathrm{Lie}(\Gamma)$.

Proof. We show that $\mathrm{LS}(\Gamma)$ is a Lie algebra. If $A, B \in \mathrm{LS}(\Gamma)$, then $\pm A$, $\pm B \in \mathrm{LS}(\Gamma)$ by Lemma 7.1. Thus $\alpha A + \beta B \in \mathrm{LS}(\Gamma)$, α, $\beta \in \mathbb{R}$, since $\mathrm{LS}(\Gamma)$ is a cone. Moreover $\pm[A, B] \in \mathrm{LS}(\Gamma)$. It follows that $\mathrm{LS}(\Gamma)$ is a Lie subalgebra of L.

Taking into account the chain $\Gamma \subset \mathrm{LS}(\Gamma) \subset \mathrm{Lie}(\Gamma)$, we conclude that $\mathrm{LS}(\Gamma) = \mathrm{Lie}(\Gamma)$. □

Theorem 7.1. *A right-invariant system $\Gamma \subset L$ is controllable on a compact connected Lie group G if and only if* $\mathrm{Lie}(\Gamma) = L$.

Proof. Apply Corollary 7.1. □

Example 7.1 ($\mathrm{SO}(3)$)**.** Let $G = \mathrm{SO}(3)$, the set of all 3×3 real orthogonal matrices with positive determinant. The Lie group G is compact and connected. Its Lie algebra $L = \mathrm{so}(3)$ is the set of all 3×3 real skew-symmetric matrices.

Take any linearly independent matrices $A_1, A_2 \in \mathrm{so}(3)$ and consider the right-invariant system $\Gamma = \{A_1, A_2\}$. Notice that the matrices A_1, A_2, and $[A_1, A_2]$ span the whole Lie algebra $\mathrm{so}(3)$. By Theorem 7.1, the system Γ is controllable. That is, any rotation in $\mathrm{SO}(3)$ can be written as the product of exponentials

$$\exp(t_N A_{i_N}) \cdots \exp(t_1 A_{i_1}), \quad t_j \geq 0, \ i_j \in \{1, 2\}, \ N \in \mathbb{N}. \tag{21}$$

The single-input right-invariant affine in control system

$$\dot{X} = (A_1 + uA_2)X, \quad u \in U \subset \mathbb{R}, \ X \in \mathrm{SO}(3) \tag{22}$$

is also controllable (for any control set U containing more than one element).

Example 7.2 $(\mathrm{SO}(n))$**.** The previous considerations are generalized to the group $G = \mathrm{SO}(n)$ of rotations of $\mathbb{R}^n$. In this case, the Lie algebra L of G is the set of all $n \times n$ skew-symmetric matrices $\mathrm{so}(n)$.

Take the matrices $A_1 = \sum_{i=1}^{n-2}(E_{i,i+1} - E_{i+1,i})$ and $A_2 = E_{n-1,n} - E_{n,n-1}$. We denote by E_{ij} the $n \times n$ matrix with the only identity in row i and column j, and all other zero entries.

It is easy to show that $\mathrm{Lie}(A_1, A_2) = \mathrm{so}(n)$. Thus, even though the group $\mathrm{SO}(n)$ is $\frac{1}{2}n(n-1)$-dimensional, the system

$$\dot{X} = (A_1 + uA_2)X, \quad X \in \mathrm{SO}(n), \ u \in U \subset \mathbb{R},$$

in which only one control is involved, is controllable (if the set of control parameters U contains at least two distinct points).

Notice that the set of pairs (A_1, A_2) such that $\mathrm{Lie}(A_1, A_2) = L$ is open and dense in $L \times L$ (this is valid for any semisimple Lie algebra L; see [22]). Thus, we can replace the matrices A_1 and A_2 by an "almost arbitrary" pair in $L \times L$.

8 Semisimple Lie groups

Definition 8.1. A subspace $I \subset L$ is called an *ideal* of a Lie algebra L if

$$[I, L] \subset I.$$

Definition 8.2. A Lie algebra L is called *simple* if it contains no proper (i.e., distinct from $\{0\}$ and L) ideals.

Definition 8.3. A Lie algebra L is called *semisimple* if it contains no nonzero Abelian ideals.

A semisimple Lie algebra is a direct sum of its simple ideals.

Definition 8.4. A Lie group G is called *simple* (resp., *semisimple*) if its Lie algebra L is simple (resp., semisimple).

The Lie groups $\mathrm{SL}(n)$ and $\mathrm{SU}(n)$ are simple; the Lie groups $\mathrm{SO}(n)$, $n \neq 4$, are simple, while $\mathrm{SO}(4)$ is semisimple.

For the controllability problem, we are interested in the case of $\mathrm{SL}(n)$ since the other two groups are compact and for them controllability is equivalent to the rank condition.

We start from an example of a control system that has the full rank and is not controllable.

Example 8.1. Let $G = \mathrm{SL}(2)$ and $\Gamma = A + \mathbb{R}B \subset \mathrm{sl}(2)$. Here A and B are traceless matrices of the form

$$\begin{aligned} A &= (a_{ij}), \qquad a_{12} > 0, \quad a_{21} > 0, \\ B &= \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \qquad b \neq 0. \end{aligned}$$

Let us show first that

$$\mathrm{Lie}(A, B) = L = \mathrm{sl}(2). \tag{23}$$

Since $\dim \mathrm{sl}(2) = 3$, we have to obtain just one element in $\mathrm{Lie}(A, B)$ linearly independent of A and B. Compute the commutator:

$$[A, B] = 2b \begin{pmatrix} 0 & -a_{12} \\ a_{21} & 0 \end{pmatrix},$$

now it is obvious that

$$\mathrm{span}(A, B, [A, B]) = \mathrm{sl}(2).$$

Equality (23) follows, i.e., the system Γ is full-rank.

In order to show that Γ is noncontrollable on $\mathrm{SL}(2)$, we prove that the bilinear system $\theta_* \Gamma$ is noncontrollable on $\mathbb{R}^2 \setminus \{0\}$. The induced system reads

$$\dot{x} = Ax + uBx, \qquad x \in \mathbb{R}^2 \setminus \{0\}, \quad u \in \mathbb{R}. \tag{24}$$

It is easy to see that the field Bx is tangent to the axes of coordinates $\{x_1 = 0\}$ and $\{x_2 = 0\}$. On the other hand, the field Ax is directed inside the first quadrant $\mathbb{R}^2_+ = \{x_1 \geq 0, \ x_2 \geq 0\}$ on its boundary. Consequently, $\mathbb{R}^2_+$ is an invariant set of the bilinear system (24). Thus the induced system $\theta_* \Gamma$ is not controllable on the homogeneous space $\mathbb{R}^2 \setminus \{0\}$, hence the right-invariant system Γ is not controllable on the Lie group $\mathrm{SL}(2)$.

The controllability problem on $\mathrm{SL}(n)$ is much harder than the one on compact Lie groups. In fact, the whole machinery of the Lie saturation on Lie groups was developed primarily for the study of controllability on $\mathrm{SL}(n)$. There are no controllability tests in this case, but there are good sufficient conditions for controllability on $\mathrm{SL}(n)$.

Theorem 8.1. *Let $G = \mathrm{SL}(n)$ and $\Gamma = A + \mathbb{R}B \subset \mathrm{sl}(n)$. Suppose that the matrices $A = (a_{ij})$ and B satisfy the conditions:*

(1) $a_{1n}a_{n1} < 0$;

(2) *the matrix A is permutation-irreducible;*

(3) $B = \mathrm{diag}(b_1, \ldots, b_n)$;

(4) $b_1 < b_2 < \cdots < b_n$;

(5) $b_i - b_j \neq b_k - b_m$ *for* $(i, j) \neq (k, m)$.

Then the system Γ is controllable on the group $\mathrm{SL}(n)$.

An $n \times n$ matrix A is called *permutation-reducible* if there exists a permutation matrix P such that

$$P^{-1}AP = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix},$$

where A_3 is a $k \times k$ matrix with $0 < k < n$. An $n \times n$ matrix is called *permutation-irreducible* if it is not permutation-reducible. Permutation-irreducible matrices are matrices having no nontrivial invariant coordinate subspaces.

Now we prove Theorem 8.1 (in the case $n = 2$ only: in the general case the proof is longer but uses essentially the same ideas).

Proof. In the case $n = 2$ we have:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad a_{12}a_{21} < 0,$$

$$B = \begin{pmatrix} -b & 0 \\ 0 & b \end{pmatrix}, \qquad b > 0.$$

Without loss of generality, we can assume that

$$a_{12} > 0, \quad a_{21} < 0,$$

in the case of opposite signs the proof is the same.

We show that

$$\mathrm{LS}(\Gamma) = \mathrm{sl}(2) = \mathrm{span}(E_{11} - E_{22}, E_{12}, E_{21}). \tag{25}$$

First of all,

$$\mathrm{LS}(\Gamma) \ni \frac{A + uB}{|u|} \rightarrow_{u \to \pm\infty} \pm B \in \mathrm{LS}(\Gamma),$$

thus

$$A, \ \pm B \in \mathrm{LS}(\Gamma).$$

That is why

$$A_t = \exp(t \,\mathrm{ad}\, B)A \in \mathrm{LS}(\Gamma), \qquad t \in \mathbb{R}.$$

Compute the matrix of the adjoint operator

$$\mathrm{ad}\, B \,:\, \mathrm{sl}(2) \to \mathrm{sl}(2), \qquad B = -b(E_{11} - E_{22}),$$

in the basis (25). We have

$$\begin{aligned} (\mathrm{ad}\, B)(E_{11} - E_{22}) &= 0, \\ (\mathrm{ad}\, B)E_{12} &= -2bE_{12}, \\ (\mathrm{ad}\, B)E_{21} &= 2bE_{21}. \end{aligned}$$

Thus the adjoint operator has the diagonal matrix

$$\mathrm{ad}\, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2b & 0 \\ 0 & 0 & 2b \end{pmatrix},$$

and its exponential is easily computed:

$$\exp(t \,\mathrm{ad}\, B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(-2bt) & 0 \\ 0 & 0 & \exp(2bt) \end{pmatrix}.$$

Further, in the basis (25)

$$A = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \end{pmatrix}, \qquad A_t = \exp(t \operatorname{ad} B)A = \begin{pmatrix} a_{11} \\ \exp(-2bt)a_{12} \\ \exp(2bt)a_{21} \end{pmatrix} \in \mathrm{LS}(\Gamma).$$

Since $\pm B = \mp b(E_{11} - E_{22}) \in \mathrm{LS}(\Gamma)$, it follows that $\pm(E_{11} - E_{22}) \in \mathrm{LS}(\Gamma)$, and we can kill the first coordinate of A_t:

$$A_t^1 = A_t - a_{11}(E_{11} - E_{22}) = \begin{pmatrix} 0 \\ \exp(-2bt)a_{12} \\ \exp(2bt)a_{21} \end{pmatrix} \in \mathrm{LS}(\Gamma).$$

We go on:

$$\mathrm{LS}(\Gamma) \ni \exp(2bt)A_t^1 = \begin{pmatrix} 0 \\ a_{12} \\ \exp(4bt)a_{21} \end{pmatrix} \to_{t\to-\infty} \begin{pmatrix} 0 \\ a_{12} \\ 0 \end{pmatrix} \in \mathrm{LS}(\Gamma).$$

Consequently,

$$\frac{1}{a_{12}} \begin{pmatrix} 0 \\ a_{12} \\ 0 \end{pmatrix} = E_{12} \in \mathrm{LS}(\Gamma).$$

Similarly,

$$\mathrm{LS}(\Gamma) \ni \exp(-2bt)A_t^1 = \begin{pmatrix} 0 \\ \exp(-4bt)a_{12} \\ a_{21} \end{pmatrix} \to_{t\to+\infty} \begin{pmatrix} 0 \\ 0 \\ a_{21} \end{pmatrix} \in \mathrm{LS}(\Gamma).$$

Then

$$\frac{1}{|a_{21}|} \begin{pmatrix} 0 \\ 0 \\ a_{21} \end{pmatrix} = -E_{21} \in \mathrm{LS}(\Gamma).$$

Summing up,

$$E_{12} - E_{21} \in \mathrm{LS}(\Gamma).$$

But this element generates a periodic one-parameter group:

$$\exp(t(E_{12} - E_{21})) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

That is why

$$\pm(E_{12} - E_{21}) \in \mathrm{LS}(\Gamma).$$

Recall that $\pm(E_{11} - E_{22}) \in \mathrm{LS}(\Gamma)$ as well. Thus

$$\pm[E_{12} - E_{21}, E_{11} - E_{22}] = \mp 2(E_{12} + E_{21}) \in \mathrm{LS}(\Gamma).$$

It follows that

$$\pm E_{12},\ \pm E_{21}, \pm(E_{11} - E_{22}) \in \mathrm{LS}(\Gamma),$$

thus $\mathrm{LS}(\Gamma) = \mathrm{sl}(2)$, and the system Γ is controllable on $\mathrm{SL}(2)$. □

There exist generalizations of the previous theorem for the case of complex spectrum of the matrix B and for general semisimple Lie groups G [10, 11, 1].

9 Solvable Lie groups

For a Lie algebra L, its *derived series* is the following descending chain of subalgebras:

$$L \supset L^{(1)} = [L, L] \supset L^{(2)} = [L^{(1)}, L^{(1)}] \supset \cdots .$$

Definition 9.1. A Lie algebra L is called *solvable* if its derived series stabilizes at zero:

$$L \supset L^{(1)} \supset L^{(2)} \supset \cdots \supset L^{(N)} = \{0\}.$$

A Lie group with a solvable Lie algebra is called *solvable*.

Exercise 9.1. Show that the triangular algebra $\mathrm{t}(n)$ is solvable.

There is a general controllability test for right-invariant systems on connected, simply connected solvable Lie groups (recall that a topological space M is called *simply connected* if any closed loop in M can be contracted to a point).

Theorem 9.1. *Let a Lie group G be connected, simply connected, and solvable. A right-invariant system $\Gamma \subset L$ is controllable iff the following two properties hold:*

(1) $\mathrm{Lie}(\Gamma) = L$ *and*

(2) Γ *is not contained in a half-space in L bounded by a subalgebra.*

If G is not simply connected, conditions (1), (2) *remain sufficient for controllability of* Γ.

We will prove only the easy part of this test — necessity. Sufficiency is highly nontrivial, its proof may be found in [13].

And necessity in Theorem 9.1 is a consequence of the following necessary controlability condition for general (not necessarily solvable) simply connected Lie groups.

Theorem 9.2. *Let G be a connected, simply connected Lie group, and let $\Gamma \subset L$. If Γ is contained in a half-space in L bounded by a subalgebra, then Γ is not controllable on G.*

Proof. Suppose that Γ is contained in a half-space $\Pi \subset L$ bounded by a subalgebra $l \subset L$, $\dim l = \dim L - 1$. We have $\Pi = \mathbb{R}_+ A + l$ for some $A \in L$. There exists a Lie subgroup $H \subset G$ with the Lie algebra l. Since G is simply connected and $\dim H = \dim G - 1$, the subgroup H is closed in G. Then the coset space

$$\{G/H = \{XH \mid X \in G\}$$

is a smooth manifold. Moreover,

$$\dim G/H = \dim G - \dim H = 1.$$

Further, since G is simply connected, its quotient G/H is simply connected as well. Summing up,

$$G/H = \mathbb{R}.$$

The quotient G/H is a homogeneous space of G: the transitive action is

$$\theta : G \times G/H \to G/H, \qquad \theta(Y, XH) = YXH.$$

In order to show that Γ is noncontrollable on G, we prove that the induced system $\theta_*\Gamma$ is noncontrollable on the homogeneous space G/H.

Denote the projection

$$\pi : G \to G/H, \qquad \pi(X) = XH.$$

For any $C \in l$ we have

$$\begin{aligned}
\theta_* C|_{\pi(\mathrm{Id})} &= \left.\frac{d}{dt}\right|_{t=0} \theta(\exp(tC), H) = \left.\frac{d}{dt}\right|_{t=0} \underbrace{\exp(tC)}_{\in H} \cdot H \\
&= \left.\frac{d}{dt}\right|_{t=0} \pi(H) = \left.\frac{d}{dt}\right|_{t=0} \pi(\mathrm{Id}) \\
&= 0.
\end{aligned}$$

That is,

$$\theta_* l|_{\pi(\mathrm{Id})} = 0.$$

Since $\Gamma \subset \Pi = \mathbb{R}_+ A + l$, then

$$\theta_*\Gamma|_{\pi(\mathrm{Id})} \subset \theta_*(\mathbb{R}_+ A + l)|_{\pi(\mathrm{Id})} = \theta_*(\mathbb{R}_+ A)|_{\pi(\mathrm{Id})} = \mathbb{R}_+ \; \theta_* A|_{\pi(\mathrm{Id})}.$$

So admissible velocities of the induced system $\theta_*\Gamma$ at $\pi(\mathrm{Id}) \in \mathbb{R}$ belong to a half-line. Thus $\theta_*\Gamma$ is not controllable on $\mathbb{R}$ and Γ is not controllable on G. $\square$

In addition to Theorem 9.1, it would be desirable to have a controllability condition with easy to verify hypotheses. We give such a condition for a subclass of solvable Lie groups.

Definition 9.2. A solvable Lie algebra is called *completely solvable* if all adjoint operators $\operatorname{ad} A$, $A \in L$, have only real eigenvalues.

Exercise 9.2. Show that the triangular algebra $\mathrm{t}(n)$ is completely solvable.

Definition 9.3. A Lie algebra is called *nilpotent* if all adjoint operators $\operatorname{ad} A$, $A \in L$, have only zero eigenvalues.

Thus any nilpotent Lie algebra is completely solvable. An example of a solvable but not completely solvable Lie algebra is provided by the Lie algebra $\mathrm{e}(2)$ of the Euclidean group of the plane.

Theorem 9.3. *Let G be a completely solvable, connected, simply connected Lie group, and let*

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \mid u_i \in \mathbb{R} \right\} \subset L.$$

The system Γ is controllable iff $\mathrm{Lie}(B_1, \ldots, B_m) = L$.

Proof. Sufficiency. We have

$$\mathrm{LS}(\Gamma) \ni \frac{A + u_i B}{|u_i|} \to_{u \to \pm\infty} \pm B_i \in \mathrm{LS}(\Gamma),$$

thus

$$\mathrm{Lie}(B_1, \ldots, B_m) \subset \mathrm{LS}(\Gamma).$$

If $\mathrm{Lie}(B_1, \ldots, B_m) = L$, then $\mathrm{LS}(\Gamma) = L$, and Γ is controllable.

Necessity is based upon the following general fact: in a completely solvable Lie algebra L, any subalgebra $l_1 \subset L$, $l_1 \neq L$, is contained in a subalgebra $l_2 \supset l_1$ such that $\dim l_2 = \dim l_1 + 1$, see [15].

Let $\mathrm{Lie}(B_1, \ldots, B_m) = l_1 \neq L$. Then there exists a codimension one subalgebra l_2 in L containing l_1:

$$l_1 \subset l_2 \subset L, \qquad \dim l_2 = \dim L - 1.$$

The system Γ is contained in a larger system:

$$\Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i \right\} \subset A + \mathrm{Lie}(B_1, \ldots, B_m) = A + l_1 \subset \mathbb{R}_+ A + l_2.$$

(1) If $A \notin l_2$, then $\Pi = \mathbb{R}_+ A + l_2$ is a half-space bounded by the subalgebbra l_2 and containing Γ. Thus Γ is not controllable.

(2) And if $A \in l_2$, then $\mathbb{R}_+ A + l_2 = l_2$ is a subalgebra containing Γ. Thus Γ is not full-rank, thus it is not controllable. □

Remarks on bibliography

The bibliography contains references of three kinds:

(1) Textbooks on geometric control theory [2, 9], and differential geometry and Lie groups theory [23].

(2) Works on controllability of right-invariant systems on Lie groups for further acquaintance with the subject [1, 3, 8, 10, 11, 12, 13, 15, 16, 17, 20, 21, 22], including a survey on the subject [18].

(3) Other works referred to in these notes.

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Index