Homework Assignment 6 in Topology I, MATH636

due to Oct 19, 2009

1. Exercises on metrizability:
   a. Let $X$ be a compact Hausdorff space. Show that $X$ is metrizable if and only if $X$ is second countable;
   b. Solve problem 5, p.29 in the text;
   c. Give an example showing that Hausdorff second countable space need not be metrizable.

2. Solve problem 1, p. 31 in the text;

3. Exercises on completely regular spaces:
   a. Solve problem 2, p. 28 in the text;
   b. Fix a subbasis $\mathcal{S}$ of a $T_1$ topological space $X$. Prove that $X$ is completely regular if and only if for each point $x \in X$ and its neighborhood $V \in \mathcal{S}$, there exists a map $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in X \setminus V$;
   c. Show that the arbitrary product of completely regular spaces is completely regular.

4. Exercises on the Tietze Extension Theorem:
   a. Show that the Tietze extension theorem implies the Urysohn lemma (i.e. if the statement of the Tietze extension theorem is true, then the Urysohn lemma is true; ignore here that the Tietze extension theorem is proved using the Urysohn lemma);
   b. Solve problem 2, p. 31 in the text;
   c. Look on the proof of the Tietze extension theorem given in the class (which coincides in essence with the proof in the text): Let $F$ be a closed set of a normal space $X$. Let $\alpha = \frac{1}{3}$. Given a continuous map $h : F \to [0, r]$, where $r > 0$ let $\tilde{h} : X \to [0, \alpha r]$ be a continuous map such that
      \[
      \tilde{h}(x) = \begin{cases} 
        0 & \text{if } x \in F \text{ and } h(x) \leq \alpha r, \\
        \alpha r & \text{if } x \in F, \text{ and } h(x) \geq (1 - \alpha)r. 
      \end{cases}
      \]
      Then for a continuous function $f : F \to [0, 1]$ define recursively $g_1 = \tilde{f}$, $g_2 = (\tilde{f} - g_1)$, $\ldots$, $g_n = \left(\tilde{f} - \sum_{i=1}^{n-1} g_i\right)$, $\ldots$. The required extension of $f$ to $X$ can be taken as $\sum_{n=1}^{\infty} g_n$. For what values of $0 < \alpha < \frac{1}{2}$ other than $\alpha = \frac{1}{3}$ (if any) does the proof of the Tietze theorem go through? Explain.